NONCOMMUTATIVE COORDINATES FOR SYMPLECTIC REPRESENTATIONS

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Abstract. We introduce coordinates on the space of Lagrangian decorated and framed representations of the fundamental group of a surface with punctures into the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. These coordinates provide a non-commutative generalization of the parametrizations of the spaces of representations into $\mathrm{SL}(2,\mathbb{R})$ given by Thurston, Penner, and Fock–Goncharov. With these coordinates, the space of framed symplectic representations provides a geometric realization of the noncommutative cluster algebras introduced by Berenstein–Retakh. The locus of positive coordinates maps to the space of decorated maximal representations. We use this to determine the homotopy type of the space of decorated maximal representations, and its homeomorphism type when $n = 2$.

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1. Introduction

In their seminal paper [10], Fock and Goncharov introduced a pair of moduli spaces, the $\mathcal{X}$-space and the $\mathcal{A}$-space, which are closely related to the variety of representations of the fundamental group of a surface $S_{g,k}$ of genus g with n-punctures into a split real simple Lie group G. They introduced explicit cluster $\mathcal{X}$-coordinates and $\mathcal{A}$-coordinates associated to an ideal triangulation of $S_{g,k}$ on these spaces. Changing the triangulation, the coordinates change by positive rational functions. Thus the locus of positive coordinates is independent of the choice of triangulation. When G is $\mathrm{SL}(2,\mathbb{R})$, the positive locus in the $\mathcal{X}$-space is closely related to the Teichmüller space, and the positive locus in the $\mathcal{A}$-space to the decorated Teichmüller space of $S_{g,k}$, and the Fock–Goncharov coordinates are extensions of Thurston's shear coordinates, respectively Penner's λ-lengths. When G is a split real group of higher rank, these moduli spaces give higher Teichmüller spaces, and the positive locus of the $\mathcal{X}$-space is closely related to the Hitchin component in the representation variety.

The set of positive representations of Fock–Goncharov and the Hitchin components account only for one family of higher Teichmüller spaces, another family is given by maximal representations into Lie groups of Hermitian type. The symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$ form essentially the only family of Lie groups that are both split real forms and of Hermitian type. In this article we generalize the work of Fock–Goncharov in the following way. We introduce two new moduli spaces, an $\mathcal{X}$-space and $\mathcal{A}$-space of representations of the fundamental group of $S_{g,k}$ into the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$, and describe non-commutative A_1-type cluster coordinates on them. We show, on the one hand, that the positive locus of the $\mathcal{X}$-space corresponds precisely to maximal representations into $\mathrm{Sp}(2n,\mathbb{R})$; we use this to determine the homotopy type of the space of maximal representations, and for $\mathrm{Sp}(4,\mathbb{R})$ also its homeomorphism type. On the other hand, we show that the $\mathcal{A}$-space gives a geometric realization of the non-commutative cluster algebras introduced by Berenstein and Retakh [2].

In Fock–Goncharov's work, an important role is played by Lusztig's total positivity, in our work, a similar role is played by positivity related to the Maslov index. As such, our work fits well in the framework of Θ-positivity, recently introduced by Guichard and Wienhard [13–15, 22], that generalizes Lusztig's total positivity and provides a unifying framework for the different higher Teichmüller spaces. In work in progress, we are extending the construction to more general Lie groups admitting a Θ-positive structure.

When the Fock–Goncharov approach is applied to the group $\mathrm{Sp}(2n,\mathbb{R})$, they define a positive locus in the space of symplectic representations. It is important to remark that the positive locus that our approach gives in the space of symplectic representations is larger than the Fock–Goncharov's one (see Section 4.6 for more details). This is because the two theories are based on two different Θ-positive structures on $\mathrm{Sp}(2n,\mathbb{R})$: respectively the one for split groups and the one for groups of Hermitian type. The perspective chosen in the present paper is the one which is suitable for describing the spaces of maximal representations.

We now describe our results in more detail.

1.1. **The pair of moduli spaces.** We introduce two moduli spaces, the space of *decorated symplectic representations* (i.e. a representation $\pi_1(S_{g,k}) \to \mathrm{Sp}(2n, \mathbb{R})$ together with a consistent choice of Lagrangian subspaces, which are fixed by peripheral elements in $\pi_1(S_{g,k})$) which serves as our $\mathcal{X}$-space, and the space of *framed symplectic representation* (i.e. $\pi_1(S_{g,k}) \to \mathrm{Sp}(2n, \mathbb{R})$ together with a consistent choice of framed Lagrangian subspaces, i.e. a Lagrangian together with a basis, which are fixed by peripheral elements in $\pi_1(S_{g,k})$) which serves as our $\mathcal{A}$-space.

Fixing an ideal triangulation $\mathcal{T}$ of $S_{g,k}$, we introduce systems of $\mathcal{X}$-coordinates, using invariant of triples, 4-tuples, and 5-tuples of Lagrangian subspaces. A system of $\mathcal{X}$-coordinates consists of a triangle invariant for each triangle, which is given by the Maslov index of the three Lagrangians associated to the vertices of the triangle, an edge invariant for every edge of the triangulation, which can be seen as a cross-ratio function of four Lagrangians, and an angle invariant, associated to each corner of a triangle, which comes from an invariant of 5-tuples of Lagrangians. We then describe in detail a map denoted by rep from the set $\mathcal{X}(\mathcal{T})$ of $\mathcal{X}$-coordinates to the space of decorated representations. A special role is played by the set $\mathcal{X}^+(\mathcal{T})$ of positive $\mathcal{X}$-coordinates, those for which the triangle invariants are equal to n, the edge invariants are just n-tuples of positive real numbers, and the angle invariants take values in $\mathrm{O}(n)$.

Theorem 1.1. *The map* rep *induces a proper surjection with generically finite fibers from $\mathcal{X}^+(\mathcal{T})$ to the space of decorated maximal representations*

Maximal representations into Lie groups of Hermitian type have been introduced in [7], and further studied in [6, 21]. All maximal representation are discrete embeddings, and spaces of maximal representations are examples of higher Teichmüller spaces.

Let us emphasize that the correspondence between positive $\mathcal{X}$-coordinates and decorated maximal representations is not a one-to-one. To every decorated maximal representation corresponds a system of positive $\mathcal{X}$-coordinates, but in general only the edge invariants are uniquely determined, the angle invariants involve some choices. We also explicitly describe the fibers of the map rep (Proposition 4.8 and Theorem 6.18) .

The $\mathcal{X}$-coordinates are more geometric, and can be used to determine the topology of the space of maximal representations, but they do not have nice algebraic properties. For example, we did not include in this paper the explicit formulas for the change of coordinates of $\mathcal{X}$-coordinates under a flip of the triangulation as they involve some unpleasant operations such as diagonalizing symmetric matrices. The $\mathcal{A}$-coordinates have better and cleaner algebraic properties.

To define the $\mathcal{A}$-coordinates on the space of framed symplectic representations, we introduce the symplectic Λ-length, which is a (complete) invariant of pairs of framed Lagrangians. Let ω denote the symplectic form, and let $(L_{\mathbf{e}}, \mathbf{e})$ $(L_{\mathbf{f}}, \mathbf{f})$ be a pair of transverse framed Lagrangians, where

$\mathbf{e} = (e_1, \dots, e_n)$ is a basis of $L_\mathbf{e}$ and $\mathbf{f} = (f_1, \dots, f_n)$ a basis of $L_\mathbf{f}$, then the symplectic Λ-length is $\Lambda_{\mathbf{e},\mathbf{f}} := (\omega(e_i, f_j))_{i,j=1,\dots,n}$. It takes values in $\mathrm{GL}(n, \mathbb{R})$, and provide a non-commutative generalization of Penner's λ-lengths (or rather their square roots), which are the special case when $n = 1$. We show that the symplectic Λ-lengths satisfy a noncommutative analogue of the Ptolemy-relation, as well as special triangle relations (which are trivially satisfied for $n = 1$). These relations have been established in greater generality for quasi-determinants in work of Berenstein and Retakh [2], and we show that in fact the symplectic Λ-length is a quasi-determinant of a 2×2 matrix over the non-commutative ring of $n \times n$-matrices. A system of $\mathcal{A}$-coordinates associates to every oriented edge the symplectic Λ-length of the two framed Lagrangians at the vertices of the edge. The non-commutative Ptolemy equation translates into an explicit formula for the changes of $\mathcal{A}$-coordinates under a flip.

Theorem 1.2. *Let $(L_1, \mathbf{v_1}), (L_2, \mathbf{v_2}), (L_3, \mathbf{v_3}), (L_4, \mathbf{v_4})$ be four pairwise transverse framed Lagrangian, which are the labels of the vertices of a quadrilateral with diagonals connecting the vertex L_2 to L_4 and the vertex L_1 to L_3 (see Figure 1.1. Let $\Lambda_{i,j}$ be the symplectic Λ-length associated to the pair $(L_i, \mathbf{v_i}), (L_j, \mathbf{v_j})$, then*

$$\Lambda_{24} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.$$

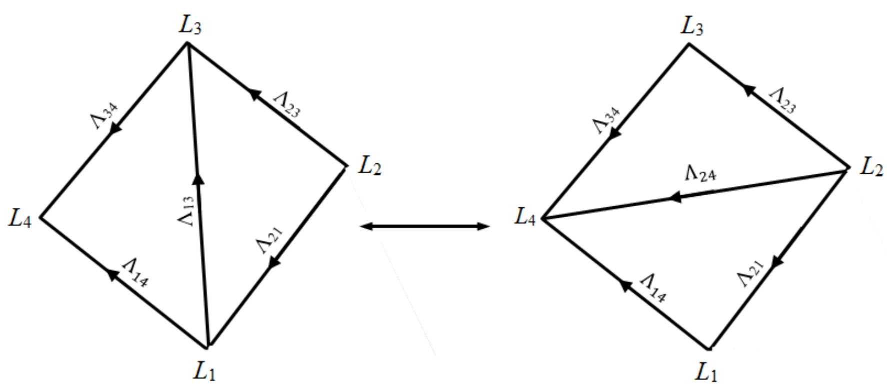

FIGURE 1.1.

We construct a natural map from $\mathcal{A}$-coordinates to $\mathcal{X}$-coordinates, which allows us to give an explicit formula for the coordinate transformation of a flip, making a local change of coordinates, see Lemma 2.20 and Proposition 8.12. In this local change of coordinates, every edge is labeled by a symmetric $n \times n$ matrix. When $n = 1$ this formula reduces to the formula for the flip in the $\mathrm{SL}(2, \mathbb{R})$-situation, and in general it is a non-commutative generalization of it. This let us view the theory of symplectic representations decorated by framed Lagrangian subspaces as a non-commutative A_1-theory. We will make this analogy even more clear for any (classical) Hermitian Lie group of tube type in forthcoming work, by showing that any such group can be realized as Sp_2 over a non-commutative ring.

This fits very well with the framework of Θ-positivity that is currently being developed by Guichard–Wienhard [13–15, 22]. There are four families of Lie groups which admit a Θ-positive structure, where Θ is a subset of the set of simple restricted roots. One is the family of split real Lie groups, the second one is the family of Hermitian Lie groups of tube type, the third family consists of the groups $\mathrm{SO}(p,q)$ with $p < q$, and the fourth is an exceptional family consisting of four groups which are real forms of real rank 4 of the complex simple Lie groups of type F_4, E_6, E_7, E_8. In the case of Hermitian Lie groups of tube type, positivity is governed by a Weyl group of type A_1, giving Θ-positivity in that case the flavor of a non-commutative A_1-theory. This is precisely what is reflected in the structure of the coordinates we define here. In forthcoming work we will define coordinates for appropriately decorated representations into $\mathrm{SO}(p,q)$, such that the positive locus corresponds to the set of Θ-positive representations.

1.2. **Topology of the space of maximal representations.** We now discuss the applications to the topology of the space of (decorated) maximal representations. Let us point out that contrary to the space of positive representations or the Hitchin component, which are contractible, the space of maximal representations has non-trivial topology. In the case of maximal representations of fundamental groups of closed surfaces, the topology of the space of maximal representations has been studied using the theory of Higgs bundles in [1, 5, 11, 12]. These techniques do not apply easily to the case of maximal representations of fundamental groups of surface with punctures, in particular since we do not fix the holonomy along peripheral curves on the surface.

Here we rely on Theorem 1.1 and the positive locus of the $\mathcal{X}$-coordinates to determine the topology of the space of maximal representations. Note that the positive locus of the $\mathcal{X}$-coordinates does not parametrize the space of decorated maximal representations, but maps surjectively to it. The fibers of this surjection are complicated to describe, because they depend on the shape of the edge invariants. However, there is a special subset of representations, for which the edge invariants are "totally degenerate", where we can describe the fibers explicitly (see Section 6.5). From this, one can deduce

Theorem 1.3. *The space of decorated maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *is homotopy equivalent to* $\mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n)$*, where the action of* $\mathrm{O}(n)$ *is by simultaneous conjugation.*

We furthermore determine the homotopy type of the space of decorated maximal representations into any connected central extension of $\mathrm{PSp}(2n,\mathbb{R})$, see Theorem 7.6.

As a corollary, we obtain a different proof of [21, Theorem 7.2.7] on the number of connected components.

Corollary 1.4. *The space of maximal representations and the space of decorated maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *have* 2^{2g+k-1} *connected components. The space of decorated maximal representations into* $\mathrm{PSp}(2n,\mathbb{R})$ *has* 2^{2g+k-1} *connected components when* n *is even; it is connected if* n *is odd.*

We can even determine the homeomorphism type of the space of maximal representations.

Theorem 1.5. *The space of decorated maximal representations into* $\mathrm{Sp}(2n,\mathbb{R})$ *is homeomorphic to the total space of a singular fibration* $\pi : \mathcal{E} \to \Delta^n$ *over the space* Δ^n *of diagonal matrices. More precisely,* $\mathcal{E}$ *is the quotient of the trivial bundle (where* $\mathrm{Sym}^+(n,\mathbb{R})$ *denotes the space of positive definite symmetric matrices)*

$$\mathrm{pr}_2 : \left(\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}\right) \times \Delta^n \to \Delta^n,$$

by the following equivalence relation: two elements (f, D) *and* (f', D') *are in the same class if and only if* $D = D'$ $(= \mathrm{pr}_2(f, D))$ *and* f *and* f' *are conjugated by an element of* $\mathrm{Stab}_{\mathrm{O}(n)}(D)$*.*

When $n = 2$, we analyze this fibration in more detail and show that all connected components except one are orbifolds, one connected component contains a non-orbifold singularity, see Section 5.2.1.

Structure of the paper: In Section 2 we introduce the invariants of Lagrangians and framed Lagrangians which are use to define coordinates. In Section 3 we introduce the spaces of decorated and framed representations, recall the definition and key properties of maximal representations. In Section 4 we introduce positive $\mathcal{X}$-coordinates, and construct the map to decorated maximal representations. The applications for the topology of the space of maximal representations are proven in Section 5.2.1. The general $\mathcal{X}$-coordinates are introduced in Section 6, and in Section 7 we generalize them to representations into central extensions of $\mathrm{PSp}(2n,\mathbb{R})$. Finally, in Section 8 we introduce $\mathcal{A}$-coordinates, describe the relations to cluster algebras, and give exact formulas for the coordinate changes under a flip of the triangulation. The Appendix contains a description of the invariants of pairs of non-degenerate symmetric bilinear forms that are used in Section 6

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2. Invariants of Lagrangian subspaces

2.1. Lagrangian Grassmannian.

We consider the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ where ω is the standard symplectic form on $\mathbb{R}^{2n}$, i.e.

$$\omega(x, y) = \sum_{i=1}^{n} x_i y_{n+i} - \sum_{i=1}^{n} x_{n+i} y_i,$$

for $x = \sum_{i=1}^{2n} x_i e_i$, $y = \sum_{i=1}^{2n} y_i e_i$ where $(e_1, \dots, e_{2n})$ is the standard basis of $\mathbb{R}^{2n}$. With respect to the standard basis, ω can be written as

$$\omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \tag{2.1}$$

Every basis of $\mathbb{R}^{2n}$ such that ω, expressed in that basis, has the form (2.1) is called a *symplectic basis*. We will usually write a symplectic basis as $(\mathbf{e}, \mathbf{f})$, where $\mathbf{e} = (e_1, \dots, e_n), \mathbf{f} = (f_1, \dots, f_n)$, and $\omega(e_i, f_j) = \delta_{ij}$.

We denote by $\mathrm{Sp}(2n, \mathbb{R})$ the *symplectic group*,

$$\mathrm{Sp}(2n, \mathbb{R}) = \{g \in \mathrm{GL}(2n, \mathbb{R}) \mid g^T \omega g = \omega\},$$

and by $\mathrm{PSp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/\{\pm \mathrm{Id}\}$ the *projective symplectic group*.

Definition 2.1. A subspace L of $\mathbb{R}^{2n}$ is called *Lagrangian* if $\dim(L) = n$ and $\omega(u, v) = 0$ for all $u, v \in L$. The set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$ is called *Lagrangian Grassmannian*, we denote this set by $\mathrm{Lag}(2n, \mathbb{R})$.

Definition 2.2. A *framed Lagrangian* is a pair $(L, \mathbf{v})$, where $L \in \mathrm{Lag}(2n, \mathbb{R})$ and $\mathbf{v}$ is a basis of L. The set of all framed Lagrangians of $(\mathbb{R}^{2n}, \omega)$ is called *framed Lagrangian Grassmannian*, we denote this set by $\mathrm{Lag}^{fr}(2n, \mathbb{R})$. The natural projection to $\mathrm{Lag}(2n, \mathbb{R})$ turns this space into a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts naturally on $\mathrm{Lag}(2n, \mathbb{R})$ and $\mathrm{Lag}^{fr}(2n, \mathbb{R})$:

$$g(L) := \{g(x) \mid x \in L\},$$
$$g(L, (v_1, \dots, v_n)) := (g(L), (g(v_1), \dots, g(v_n))).$$

These actions are transitive, hence the spaces $\mathrm{Lag}(2n, \mathbb{R})$ and $\mathrm{Lag}^{fr}(2n, \mathbb{R})$ are homogeneous spaces over the symplectic group. To better see this structure, consider the stabilizers of a point:

$$P = \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(L), \tag{2.2}$$
$$U = \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}((L, v)). \tag{2.3}$$

The group P is a parabolic subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, and $U \subset P$ is its unipotent subgroup. As homogeneous spaces, we have

$$\mathrm{Lag}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/P,$$
$$\mathrm{Lag}^{fr}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R})/U.$$

Anyway, the action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathrm{Lag}(2n, \mathbb{R})$ is not effective, it has kernel $\{\pm \mathrm{Id}\}$. The actual group of symmetries of $\mathrm{Lag}(2n, \mathbb{R})$ is the projective symplectic group $\mathrm{PSp}(2n, \mathbb{R})$.

Definition 2.3. Two Lagrangians $L_1, L_2 \in \mathrm{Lag}(2n, \mathbb{R})$ are called *transverse* if $L_1 \oplus L_2 = \mathbb{R}^{2n}$.

We now describe charts for $\mathrm{Lag}(2n,\mathbb{R})$. Since we will work in these charts regularly, we describe them and the coordinate changes in detail. Given a Lagrangian L_∞, we denote by $\mathcal{U}_{L_\infty}$ the subset of $\mathrm{Lag}(2n,\mathbb{R})$ consisting of all the Lagrangians transverse to L. This is an open dense subset of $\mathrm{Lag}(2n,\mathbb{R})$. Fixing a Lagrangian $L_0 \in \mathcal{U}_{L_\infty}$ any other Lagrangian $L \in \mathcal{U}_{L_\infty}$ is the graph of a linear map $L_{L_0\to L_\infty} \colon L_0 \to L_\infty$, i.e. for each $v \in L_0$, $L_{L_0\to L_\infty}(v)$ is the unique element in L_∞ such that $v + L_{L_0\to L_\infty}(v) \in L$. So we define If L is also transverse to L_0, this map, which we denote just by L if there is no danger of confusion, is a linear isomorphism.

We will often use an explicit matrix expression for this linear map. If we choose $\mathbf{e} = (e_1,\dots,e_n)$ a basis of L_0, there exists a unique basis $\mathbf{f} = (f_1,\dots,f_n)$ of L_∞ such that $(\mathbf{e},\mathbf{f})$ is a symplectic basis. Given a symplectic basis $(\mathbf{e},\mathbf{f})$ will more generally write then

$$\begin{aligned} L_{\mathbf{e}} &:= \mathrm{Span}(\mathbf{e}),\\ L_{\mathbf{f}} &:= \mathrm{Span}(\mathbf{f}). \end{aligned}$$

We write $[L_{L_{\mathbf{e}}\to L_{\mathbf{f}}}]_{\mathbf{e},\mathbf{f}}$ for the matrix of the map $L_{L_{\mathbf{e}}\to L_{\mathbf{f}}}$ with respect to the bases $\mathbf{e},\mathbf{f}$. It is easy to check that this matrix is symmetric. The linear map L and its matrix $[L_{L_{\mathbf{e}}\to L_{\mathbf{f}}}]_{\mathbf{e},\mathbf{f}}$ will be used often in this paper.

We thus have a map

$$\Psi_{(\mathbf{e},\mathbf{f})} \colon \mathcal{U}_{L_{\mathbf{f}}} \ni L \to [L_{L_{\mathbf{e}}\to L_{\mathbf{f}}}]_{\mathbf{e},\mathbf{f}} \in \mathrm{Sym}(n,\mathbb{R})$$

This map is a homeomorphism to the vector space of symmetric matrices. To see that it is invertible, the inverse map is given by the formula

$$L_{\mathbf{e},\mathbf{f}}(A) := L = \mathrm{Span}(\mathbf{e} + \mathbf{f}A)$$

The set

$$\{(\mathcal{U}_{L_{\mathbf{f}}}, \Psi_{(\mathbf{e},\mathbf{f})}) \mid (\mathbf{e},\mathbf{f}) \text{ symplectic basis } \}$$

is a manifold atlas for the space $\mathrm{Lag}(2n,\mathbb{R})$.

Remark 2.4. We can write the transition functions of this atlas. Assume $(\mathbf{e},\mathbf{f})$ and $(\mathbf{e}',\mathbf{f}')$ are two symplectic bases. There is a unique symplectic matrix $B \in \mathrm{Sp}(2n,\mathbb{R})$ such that $(\mathbf{e}',\mathbf{f}') := (\mathbf{e},\mathbf{f})B^{-1}$. Write B as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}),$$

where the B_{ij} are $n\times n$ matrices. For every $L \in \mathcal{U}_{L_f} \cap \mathcal{U}_{L_{f'}}$, denote by

$$\begin{aligned} A &:= \Psi_{(\mathbf{e},\mathbf{f})}(L)\\ A' &:= \Psi_{(\mathbf{e}',\mathbf{f}')}(L) \end{aligned}$$

Then

$$A' = (B_{11} + B_{12}A)^{-1}(B_{21} + B_{22}A) \in \mathrm{Sym}(n,\mathbb{R}). \tag{2.4}$$

Remark 2.5. Formula (2.4) also represents the action of the matrix B on $\mathrm{Lag}(2n,\mathbb{R})$, when restricted to a coordinate chart $\mathcal{U}_{L_f}$: for a Lagrangian L such that both $L, B(L) \in \mathcal{U}_{L_f}$,

$$A := \Psi_{(\mathbf{e},\mathbf{f})}(L)$$
$$A^B := \Psi_{(\mathbf{e},\mathbf{f})}(B(L))$$

we have

$$A^B = (B_{11} + B_{12}A)^{-1}(B_{21} + B_{22}A) \in \mathrm{Sym}(n,\mathbb{R}).$$

In fact the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$ is formally similar to the action by Möbius transformations of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{CP}^1$ (which is the case $n = 1$).

The action of $\mathrm{Sp}(2n,\mathbb{R})$ on pairs of transverse Lagrangians is transitive, but the action of $\mathrm{Sp}(2n,\mathbb{R})$ on triples, quadruples and 5-tuples of pairwise transverse Lagrangians is not transitive any more. We will now describe invariants of such tuples of Lagrangians, which will lie the foundation for the rest of the paper.

Similarly, the action of $\mathrm{Sp}(2n,\mathbb{R})$ on pairs $((L,v),L')$, where $(L,v) \in \mathrm{Lag}^{fr}(2n,\mathbb{R})$, $L' \in \mathrm{Lag}(2n,\mathbb{R})$ and L, L' are transverse, is transitive and free. But when we consider pairs $(L,v),(L',v') \in \mathrm{Lag}^{fr}(2n,\mathbb{R})$, the action is not transitive any more, and we describe invariants of such pairs.

2.2. **Maslov index.**

In this section we review properties of the Maslov index of three pairwise transverse Lagrangians, for a more general discussion we refer the reader to [18].

Let L_1, L_2, L_3 be three pairwise transverse Lagrangians. As in the previous section, we consider the linear map $L_{3L_1\to L_2}$. When this does not cause confusion, we will denote the linear map just by L_3.

Using the symplectic form ω, we can define a bilinear form β_3 on L_1 in the following way: for $v_1, v_2 \in L_1$

$$\beta_3(v_1, v_2) := \omega(v_1, L_3(v_2)).$$

We also denote the bilinear form β_3 by $[L_1, L_3, L_2]$.

Proposition 2.6. *The bilinear form $\beta_3 = [L_1, L_3, L_2]$ is non degenerate and symmetric.*

Proof. Since $L_3(v) + v \in L_3$ for all $v \in V_1$,

$$0 = \omega(L_3v + v, L_3w + w) = \omega(L_3v, w) + \omega(v, L_3w).$$

Therefore,

$$\beta_3(v,w) = \omega(v, L_3w) = -\omega(L_3v, w) = \omega(w, L_3v) = \beta_3(w,v)$$

The form β_3 is non-degenerate because L_3 is a linear isomorphism between two transverse Lagrangians L_1 and L_2, i.e. $\omega|_{L_1\times L_2}$ is non-degenerate. $\square$

We will denote the signature of β_3 by

$$\mathrm{sgn}(\beta_3) = (p, q),$$

where p is the dimension of a maximal subspace of L_1 on which β_3 is positive definite and q is the dimension of a maximal subspace of L_1 on which β_3 is negative definite. They satisfy $p + q = n$. We will also sometimes express the signature as

$$\mathrm{dsgn}(\beta_3) = p - q \in \{-n, -n+2, \dots, n-2, n\}.$$

Definition 2.7. The *Maslov index* of the triple of Lagrangians (L_1, L_3, L_2) is the signature $\mathrm{dsgn}([L_1, L_3, L_2])$ and denoted by $\mu(L_1, L_3, L_2)$.

For $n = 1$, the three Lagrangians (L_1, L_3, L_2) correspond to distinct points in the circle $\mathbb{RP}^1$. The Maslov index is 1 if the three points are cyclically ordered, and it is -1 if they are in the reverse cyclic order.

Proposition 2.8 (Properties of Maslov index)**.** *The Maslov index*

- *is invariant under the action of* $\mathrm{Sp}(2n, \mathbb{R})$ *on* $\mathrm{Lag}(2n, \mathbb{R})$*;*
- *is anti-symmetric when two of its variables are exchanged;*
- *satisfies the cocycle relation, i.e. for all pairwise transverse* $L_1, L_2, L_3, L_4 \in \mathrm{Lag}(2n, \mathbb{R})$

$$\mu(L_1, L_2, L_3) - \mu(L_1, L_2, L_4) + \mu(L_1, L_3, L_4) - \mu(L_2, L_3, L_4) = 0$$

- *the group* $\mathrm{Sp}(2n, \mathbb{R})$ *acts transitively on the set of triples of pairwise transverse Lagrangians with the same Maslov index, i.e.* $\mathrm{Sp}(2n, \mathbb{R})$*-orbits of pairwise transverse triples of Lagrangians are in 1-1 correspondence with the Maslov indices.*

2.3. **Cross ratio.** Let L_1, L_2, L_3, L_4 be four Lagrangians such that L_3 and L_4 are transverse to L_1 and L_2. We use the linear isomorphisms $L_3 \colon L_1 \to L_2$ and $L_4 \colon L_2 \to L_1$ to introduce the map

$$[L_1, L_3, L_2, L_4] := L_4 \circ L_3 \colon L_1 \to L_1$$

which is a linear automorphism of L_1.

Definition 2.9. The map

$$[L_1, L_3, L_2, L_4] \colon L_1 \to L_1$$

is called the *cross ratio* of the 4-tuple of Lagrangians (L_1, L_3, L_2, L_4).

For related invariants of 4 Lagrangians, see [3, 4, 16, 20]. For $n = 1$, the cross ratio is a linear map from a line to itself. This is just the multiplication by a scalar, which is exactly the cross ratio of four lines in $\mathbb{R}^2$ in the classical sense.

Proposition 2.10 (Properties of cross ratio)**.**

- *The cross ratio is equivariant under the action of* $\mathrm{Sp}(2n, \mathbb{R})$ *on* $\mathrm{Lag}(2n, \mathbb{R})$*.*

- $[L_1, L_3, L_2, L_4] = [L_1, L_4, L_2, L_3]^{-1}$;
  $[L_1, L_3, L_2, L_4] = L_3^{-1} \circ [L_2, L_4, L_1, L_3] \circ L_3$.
- *The group* $\mathrm{Sp}(2n, \mathbb{R})$ *acts transitively on quadruples of pairwise transverse Lagrangians having conjugate cross ratios, i.e. the* $\mathrm{Sp}(2n, \mathbb{R})$*-orbits of pairwise transverse quadruples of Lagrangians are in 1-1 correspondence with the conjugacy classes of cross ratios.*

Proposition 2.11. *The cross ratio* $B := [L_1, L_3, L_2, L_4]$ *is a symmetric linear map with respect to the bilinear forms* $[L_1, L_3, L_2]$ *and* $[L_1, L_4, L_2]$.

Proof. Let $\beta_3 = [L_1, L_3, L_2]$ and $\beta_4 = [L_2, L_4, L_1]$ be a symmetric bilinear form on L_2. Let $v, w \in L_1$. Then:

$$\beta_3(Bv, w) = \omega(L_4 L_3 v, L_3 w) = -\omega(L_3 w, L_4 L_3 v) =$$
$$= -\beta_4(L_3 w, L_3 v) = -\beta_4(L_3 v, L_3 w) = -\omega(L_3 v, L_4 L_3 w) =$$
$$= \omega(L_4 L_3 w, L_3 v) = \beta_3(Bw, v) = \beta_3(v, Bw).$$ □

Corollary 2.12. *If* $[L_1, L_3, L_2]$ *and* $[L_2, L_4, L_1]$ *are positive definite, then* $-[L_1, L_3, L_2, L_4]$ *is diagonalizable with positive eigenvalues.*

Proof. We set as before $\beta_3 = [L_1, L_3, L_2]$ and $\beta_4 = [L_2, L_4, L_1]$. Let $\mathbf{e}$ be a basis of L_1 such that $[\beta_3]_{\mathbf{e}} = \mathrm{Id}$ and $[B]_{\mathbf{e}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We take the unique basis $\mathbf{f}$ of L_2 such that $\omega(\mathbf{e}, \mathbf{f}) = \mathrm{Id}$. Then $L_3(\mathbf{e}) = \mathbf{f}$ and $[L_3]_{\mathbf{e},\mathbf{f}} = \mathrm{Id}$.

In the basis $\mathbf{f}$ the bilinear form β_4 is diagonal because for every two basis vectors f_i, f_j

$$\beta_4(f_i, f_j) = \omega(f_i, L_4(f_j)) = \omega(L_3 L_3^{-1}(f_i), L_4 L_3 L_3^{-1}(f_j)) =$$
$$= \omega(L_3 e_i, B e_j) = -\omega(B e_j, L_3 e_i) = -\beta_3(B e_j, e_i) = \lambda_i \delta_{ij}$$

Since β_4 is positive definite, we have $\lambda_i > 0$ for all i. □

2.4. **Angles.** We will also make use of invariants of five Lagrangians, here we describe it in the simplest case, when all the Maslov indices are maximal. For the general version of this invariant, see Section 6.1. Let $L_1, \dots, L_5$ be pairwise-transverse Lagrangians, which we will think as the vertices of a pentagon, as in Figure 2.1. Assume that

$$\mu(L_1, L_3, L_2) = \mu(L_2, L_4, L_1) = \mu(L_1, L_5, L_3) = n.$$

The bilinear forms $\beta_3 = [L_1, L_3, L_2]$ and $\beta_4 = [L_2, L_4, L_1]$ are positive definite, therefore, by Corollary 2.12, there exists a basis $\mathbf{e_1}$ of L_1 such that $[\beta_3]_{\mathbf{e_1}} = \mathrm{Id}$ and $[L_1, L_3, L_2, L_4]_{\mathbf{e_1}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n > 0$.

We can do the same for the quadruple (L_3, L_2, L_1, L_5) and find a basis $\mathbf{g}$ of L_3 such that the bilinear forms $[L_3, L_2, L_1]_g = \mathrm{Id}$ and $-[L_3, L_2, L_1, L_5]_{\mathbf{g}} = \operatorname{diag}(\mu_1, \dots, \mu_n)$ with $\mu_1 \geq \dots \geq \mu_n > 0$.

We take the unique basis $\mathbf{e_2}$ on L_1 such that $\omega(\mathbf{g}, \mathbf{e_2}) = \mathrm{Id}$. In the basis $\mathbf{e_2}$ of L_1 we have

$$[\beta_3]_{\mathbf{e_2}} = [L_1, L_2, L_3]_{\mathbf{e_2}} = [L_3, L_2, L_1]_{\mathbf{g}} = \mathrm{Id}\,.$$

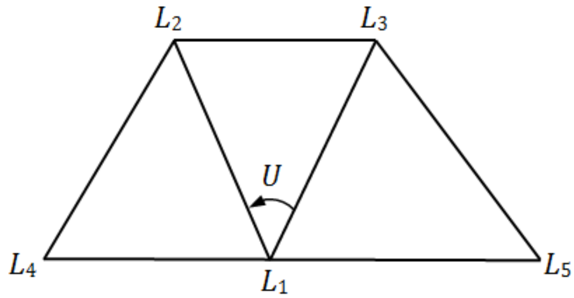

FIGURE 2.1.

Let $U \in \mathrm{O}(n)$ be the change-of-basis matrix from the basis $\mathbf{e_2}$ to the basis $\mathbf{e_1}$. We will call this matrix an inner *angle* in the pentagon of Lagrangians $(L_1, L_4, L_2, L_3, L_5)$ (see Figure 2.1).

The matrix U is not uniquely defined because the bases $\mathbf{e_1}$ and $\mathbf{g}$ are not unique. In general, U is only well defined as an element of the double coset space $\mathrm{Stab}_1 \diagdown \mathrm{O}(n)/\,\mathrm{Stab}_2$, where

$$\mathrm{Stab}_1 := \{A \in \mathrm{O}(n) \mid \; A\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)A^T = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\},$$

$$\mathrm{Stab}_2 := \{A \in \mathrm{O}(n) \mid \; A\,\mathrm{diag}(\mu_1, \ldots, \mu_n)A^T = \mathrm{diag}(\mu_1, \ldots, \mu_n)\}.$$

We denote by $[L_1, L_5, L_3, L_2, L_4]$ the class of U in $\mathrm{Stab}_1 \diagdown \mathrm{O}(n)/\,\mathrm{Stab}_2$. If bases $\mathbf{e_1}$ and $\mathbf{e_2}$ are chosen as above, we will write

$$U =: [L_1, L_5, L_3, L_2, L_4]_{e_1,e_2}.$$

2.5. **Symplectic Λ-lengths.** In this subsection we introduce an invariant of two transverse framed Lagrangians. Since this invariant is closely related to Penner's λ-lengths in the case when $n = 1$, we call the it the symplectic Λ-length.

Definition 2.13. The *symplectic* Λ*-length* of a pair of two framed Lagrangians $(L, \mathbf{v}), (M, \mathbf{w}) \in \mathrm{Lag}^{fr}(2n, \mathbb{R})$ is the $n \times n$ matrix

$$\Lambda_{(L,\mathbf{v}),(M,\mathbf{w})} := \omega(\mathbf{v}, \mathbf{w}) = (\omega(v_i, w_j)).$$

To simplify the notation, when $\{(L_i, v_i) \mid i \in I\}$ is a set of framed Lagrangians, we will write

$$\Lambda_{ij} := \Lambda_{(L_i,\mathbf{v_i}),(L_j,\mathbf{v_j})}, \text{ for } i, j \in I.$$

Proposition 2.14. *For all* $(L_1, \mathbf{v_1}), (L_2, \mathbf{v_2}) \in \mathrm{Lag}^{fr}(2n, \mathbb{R})$*, we have*

(1) $\Lambda_{12} = -\Lambda_{21}^T$.
(2) L_1 *and* L_2 *are transverse if and only if* $\det \Lambda_{12} \neq 0$.
(3) If L_1 *and* L_2 *are transverse, then*

$$[\omega]_{\mathbf{v_1},\mathbf{v_2}} = \begin{pmatrix} 0 & \Lambda_{12} \\ \Lambda_{21} & 0 \end{pmatrix}.$$

The symplectic Λ-lengths generalize Penner's λ-lengths for the decorated Teichmüller space: In [19] he considers the space $\mathbb{R}^3$ with a symmetric bilinear form b of signature $(2,1)$. A *decorated cusp* is a pair (I,w), where $I \subset \mathbb{R}^3$ is an isotropic line and $w \in I$. The *λ-length* of two decorated cusps (w_1,I_1) and (w_2,I_2) is given by $b(w_1,w_2)$.

Lemma 2.15. *When $n=1$, the symplectic Λ-length is a square root of Penner's λ-length*

Proof. For $n=1$, a pair of framed Lagrangians is given by two lines $L_1, L_2 \subset \mathbb{R}^2$ with a choice of vectors $v_1, \in L_1, v_2 \in L_2$. The symplectic Λ-lengths are $\omega(v_1,v_2) = -\omega(v_2,v_1)$. The relation to Penner's λ-length is given by considering $\mathbb{R}^3$ as the space of order 2 symmetric tensors of $\mathbb{R}^2$, where the bilinear form b induced by ω is symmetric of signature $(2,1)$. The tensor products $L_1 \otimes L_1$ and $L_2 \otimes L_2$ are isotropic lines, with choice of vectors $v_1 \otimes v_1$ and $v_2 \otimes v_2$. The Λ-length is

$$b(v_1 \otimes v_1, v_2 \otimes v_2) = \omega(v_1,v_2)^2.$$

$\square$

2.6. **Symplectic Λ-lengths and quasi-determinants.**

2.6.1. *Ptolemy Equation, Exchange and triangle relations.* Penner's λ-lengths satisfy the famous Ptolemy equation. Given four isotropic vectors w_1, w_2, w_3, w_4 in $\mathbb{R}^3$ in general position, $b(w_2,w_4)b(w_1,w_3) = b(w_2,w_3)b(w_1,w_4) + b(w_1,w_2)b(w_3,w_4)$ (see Figure 2.2). Our symplectic Λ-lengths satisfy a non-commutative version of the Ptolemy equation. We also call this identity the exchange relation. Moreover, they satisfy a triangle or hexagon relation, which are trivial in Penner's case.

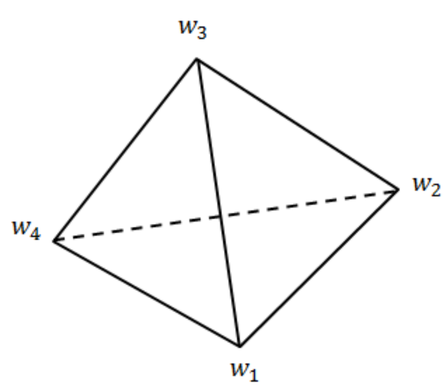

FIGURE 2.2.

Lemma 2.16. *Let $(L_1, \mathbf{v_1})$, $(L_2, \mathbf{v_2})$ be two transverse framed Lagrangians. Consider a third framed Lagrangian $(L_3, \mathbf{v_3})$, then there exist matrices*

$\Pi_3^1, \Pi_3^2 \in \mathrm{Mat}(n \times n, \mathbb{R})$ *such that*

$$\mathbf{v_3} = \mathbf{v_1}\Pi_3^1 + \mathbf{v_2}\Pi_3^2.$$

Then

$$\Pi_3^1 = \Lambda_{21}^{-1}\Lambda_{23}, \quad \Pi_3^2 = \Lambda_{12}^{-1}\Lambda_{13}.$$

Proof.

$$\Lambda_{13} = \omega(\mathbf{v_1}, \mathbf{v_3}) = \omega(\mathbf{v_1}, \mathbf{v_1}\Pi_3^1 + \mathbf{v_2}\Pi_3^2) = \omega(\mathbf{v_1}, \mathbf{v_2}\Pi_3^2) = \Lambda_{12}\Pi_3^2,$$

so

$$\Lambda_{12}^{-1}\Lambda_{13} = \Pi_3^2. \qquad \square$$

Proposition 2.17 (Ptolemy equation - Exchange relation)**.** *Let* $(L_i, \mathbf{v_i})$, $i \in \{1, 2, 3, 4\}$ *be four pairwise transverse framed Lagrangians. Then*

$$\Lambda_{24} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.$$

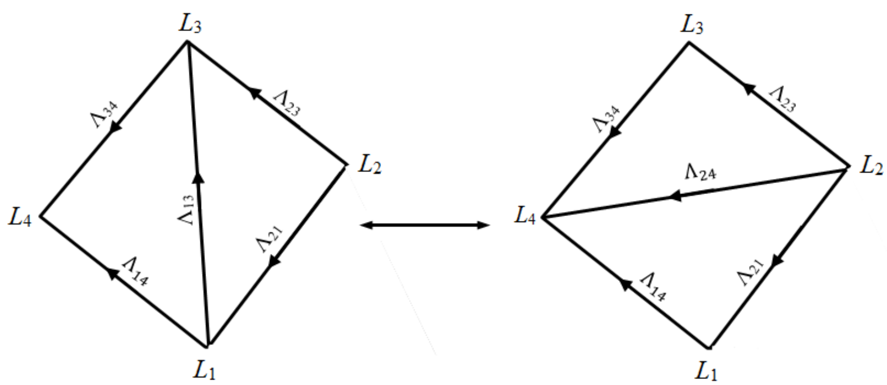

FIGURE 2.3.

Proof. Using Lemma 2.16, we have

$$\Lambda_{24} = \omega(\mathbf{v_2}, \mathbf{v_4}) = \omega(\mathbf{v_1}\Lambda_{31}^{-1}\Lambda_{32}, \mathbf{v_3}\Lambda_{13}^{-1}\Lambda_{14}) + \omega(\mathbf{v_3}\Lambda_{13}^{-1}\Lambda_{12}, \mathbf{v_1}\Lambda_{31}^{-1}\Lambda_{34}) =$$
$$= (\Lambda_{31}^{-1}\Lambda_{32})^T\Lambda_{13}\Lambda_{13}^{-1}\Lambda_{14} + (\Lambda_{13}^{-1}\Lambda_{12})^T\Lambda_{31}\Lambda_{31}^{-1}\Lambda_{34} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.$$

$\square$

Corollary 2.18 (Triangle relation)**.**

$$\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{32} = 0,$$

$$\Lambda_{32}^{-1}\Lambda_{31}\Lambda_{21}^{-1}\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12} = -1.$$

2.6.2. *Relation to quasideterminants.* Algebras satisfying the exchange and triangle relations were studied in [2]. They appear there as algebras of quasideterminants over an algebra R which is, in general, non-commutative. More precisely, they consider the R-algebra A which is freely generated by elements $A_0 = \{x_i, x_i^{-1}\}_{i\in I}$ for some finite index set I. Given $\{a_{ij} \mid i \in \{1,2\}, j \in \{1,\dots,n\}\} \subseteq A_0$ for some n, a *quasi-determinant* is an expression

$$y_{ij} := \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix} := a_{2j} - a_{2i}a_{1i}^{-1}a_{1j} \in A.$$

For quasi-determinants $\{y_{ij} \mid i \in \{1,2\}, j \in \{1,\dots,n\}\}$, the exchange and triangle relations hold.

The symplectic Λ-length can be seen as quasi-determinants over the algebra of $n \times n$ matrices. Let $(L, \mathbf{v})$ be a framed Lagrangian. The basis $\mathbf{v}$ can be represented by a $n \times 2n$-matrix $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ in the standard symplectic basis of $(\mathbb{R}^{2n}, \omega)$, where v^1, v^2 are $n \times n$-matrices. We can assume that v^1 is invertible. With this notation, given two framed Lagrangians $(L, \mathbf{v})$ and $(M, \mathbf{w})$ we can write:

$$\Lambda_{(L,\mathbf{v}),(M,\mathbf{w})} = \omega(\mathbf{v}, \mathbf{w}) = (v^1)^T w^2 - (v^2)^T w^1 = (v^1)^T (w^2 - ((v^1)^{-1})^T (v^2)^T w^1)$$

Since

$$0 = \omega(\mathbf{v}, \mathbf{v}) = (v^1)^T v^2 - (v^2)^T v^1,$$
$$((v^1)^{-1})^T (v^2)^T = v^2 (v^1)^{-1},$$

we get

$$\Lambda_{(L,\mathbf{v}),(M,\mathbf{w})} = (v^1)^T (w^2 - v^2 (v^1)^{-1} w^1) = (v^1)^T \begin{vmatrix} v^1 & w^1 \\ v^2 & \boxed{w^2} \end{vmatrix}.$$

In fact, later we will use symplectic Λ-lengths to introduce $\mathcal{A}$-type coordinates on the space of framed representations. These $\mathcal{A}$-type coordinates then provide a geometric avatar of the non-commutative cluster structure introduced by Berenstein–Retakh [2]. This will be addressed in detail in Section 8.4.

2.7. **Symplectic Λ-lengths, Maslov index and cross-ratios.** We have seen that triples and 4-tuples of pairwise transverse Lagrangians have invariants, the Maslov index and the cross ratio. If we choose bases for all of the Lagrangian subspaces, the Maslov index and the matrix of the cross ratio can be expressed in terms of the symplectic Λ-lengths.

Lemma 2.19. *Let $(L_i, \mathbf{v_i}) \in \mathrm{Lag}^{fr}(2n, \mathbb{R})$, with $i \in \{1,2,3\}$, be three pairwise transverse framed Lagrangians. Then the matrix $\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{21}$ is symmetric and the Maslov index is given by its signature:*

$$\mu(L_1, L_3, L_2) = \mathrm{dsgn}(\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{21}).$$

Proof. The matrix $\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{21}$ is symmetric because of the triangle relation and the property $\Lambda_{ij} = -\Lambda_{ji}^T$.

Let $\mathbf{v_2}'$ be a basis of L_2 dual to $\mathbf{v_1}$. By Lemma 2.16, the linear map $L_3\colon L_1 \to L_2$ in bases $(\mathbf{v_1}, \mathbf{v_2}')$ is given by the matrix $\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{21}$. But by definition, the Maslov index is the signature of the map L_3 seen as a bilinear form on L_1. □

Lemma 2.20 (Cross ratio in terms of symplectic Λ-lengths)**.** *Let $(L_i, \mathbf{v_i}) \in \mathrm{Lag}^{fr}(2n, \mathbb{R})$, with $i \in \{1, 2, 3, 4\}$, be four pairwise transverse framed Lagrangians. Then*

$$[L_1, L_2, L_3, L_4]_{\mathbf{v_1}} = \Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31} = \Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21},$$

where $[L_1, L_2, L_3, L_4]_{\mathbf{v_1}}$ denotes the cross-ratio expressed in the basis $\mathbf{v_1}$.

Proof. Let $w \in L_2$ then $w = \mathbf{v_2}a$ where $a = (a_1, \ldots, a_n)^T$. Then by Lemma 2.16

$$w = \mathbf{v_1}\Lambda_{31}^{-1}\Lambda_{32}a + \mathbf{v_3}\Lambda_{13}^{-1}\Lambda_{12}a,$$

where $\mathbf{v_1}\Lambda_{31}^{-1}\Lambda_{32}a \in L_1$, $\mathbf{v_3}\Lambda_{13}^{-1}\Lambda_{12}a \in L_3$. Therefore, the map L_2 maps $\mathbf{v_1}\Lambda_{31}^{-1}\Lambda_{32}a$ to $\mathbf{v_3}\Lambda_{13}^{-1}\Lambda_{12}a$. If we denote $b := \Lambda_{31}^{-1}\Lambda_{32}a$, then $a = \Lambda_{32}^{-1}\Lambda_{31}b$ and

$$L_2(\mathbf{v_1}b) = \mathbf{v_3}\Lambda_{13}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31}b.$$

Using the triangle relation we get the following:

$$[L_2]_{\mathbf{v_1},\mathbf{v_3}} = \Lambda_{13}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31} = -\Lambda_{13}^{-1}\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{21} = \Lambda_{23}^{-1}\Lambda_{21}.$$

We get the same for $L_4\colon L_3 \to L_1$:

$$[L_4]_{\mathbf{v_3},\mathbf{v_1}} = \Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{13} = -\Lambda_{41}^{-1}\Lambda_{43}.$$

Therefore, on one hand

$$[L_1, L_2, L_3, L_4]_{\mathbf{v_1}} = \Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{13}\Lambda_{13}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31} = \Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31},$$

on the other hand

$$[L_1, L_2, L_3, L_4]_{\mathbf{v_1}} = \Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21}.$$ □

3. Representation varieties

One goal of this article is to give a parametrization of spaces of representations of the fundamental group of a punctured surface into $\mathrm{Sp}(2n, \mathbb{R})$, which can be viewed as a non-commutative generalization of the parametrization of representations into $\mathrm{SL}(2, \mathbb{R})$ by Thurston and Penner coordinates. We will in fact not directly parameterize the representation variety, but an extension of it, which we call decorated or framed representations.

3.1. **Representation spaces.** Let S be a punctured surface of genus g with $k > 0$ punctures. We assume that the Euler characteristic $\chi(S)$ of S is negative. In this case the fundamental group $\pi_1(S)$ of S is free with $2g + k - 1 = |\chi(S)| + 1 \geq 2$ generators.

Definition 3.1. An element $g \in \pi_1(S)$ is called *peripheral* if g is freely homotopic to a loop contained in an arbitrarily small neighborhood of a puncture. We denote by $\pi_1^{per}(S)$ the subset of $\pi_1(S)$ containing all peripheral elements. Since we consider only punctured surfaces, $\pi_1^{per}(S) \neq \varnothing$.

By $\mathrm{Hom}(\pi_1(S), G)$ we denote the set of all representations of the fundamental group $\pi_1(S)$ of the surface S into some Lie group G. The group G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation.

Definition 3.2. The quotient space

$$\mathrm{Rep}(\pi_1(S), G) := \mathrm{Hom}(\pi_1(S), G)/G$$

is called the *moduli space of representations*. We denote by $[\rho]$ the class in $\mathrm{Rep}(\pi_1(S), G)$ of the representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$.

Remark 3.3. The action of G on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation is not proper, hence the quotient is, in general, not Hausdorff. The action is proper on the subset of reductive representations, which has an Hausdorff quotient, usually called the character variety. In this paper, it is more natural to consider the quotient of all representations, and to deal with a quotient space which is not Hausdorff.

Definition 3.4. A representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ will be called *peripherally parabolic* if for every $g \in \pi_1^{per}(S)$, the matrix $\rho(g)$ lies in a subgroup conjugate to P (see Formula (2.2)).

In other words, a representation is parabolic if and only if every peripheral element leaves invariant a Lagrangian in $(\mathbb{R}^{2n}, \omega)$. We will denote by $\mathrm{Hom}^P(\pi_1(S), G)$ the subset of $\mathrm{Hom}(\pi_1(S), G)$ consisting of peripherally parabolic representations.

Definition 3.5. The quotient space

$$\mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) := \mathrm{Hom}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))/\,\mathrm{Sp}(2n, \mathbb{R})$$

is called the *moduli space of peripherally parabolic representations.*

Remark 3.6. The space $\mathrm{Rep}(\pi_1(S), G)$ does not depend very much on the surface S, because it depends only on $\pi_1(S)$, and there are several surfaces with the same fundamental group. For this reason, it is not easy to study this space using topological decompositions of S. In the space $\mathrm{Rep}^P(\pi_1(S), G)$ however we put conditions on the peripheral elements in $\pi_1(S)$, and thus it depends on and is more closely related to the topology of S.

Definition 3.7. A representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ will be called *peripherally unipotent* if for every $g \in \pi_1^{per}(S)$, the matrix $\rho(g)$ lies in a subgroup conjugate to U (see Formula (2.3)).

In other words, a representation is peripherally unipotent if and only if every peripheral element leaves invariant a framed Lagrangian in $(\mathbb{R}^{2n},\omega)$. We will denote by $\mathrm{Hom}^U(\pi_1(S),G)$ the subset of $\mathrm{Hom}(\pi_1(S),G)$ consisting of peripherally unipotent representations.

Definition 3.8. The quotient space

$$\mathrm{Rep}^U(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) := \mathrm{Hom}^U(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))/\,\mathrm{Sp}(2n,\mathbb{R})$$

is called the *moduli space of peripherally unipotent representations.*

3.2. **Decorated representations.** For a peripherally parabolic representation there might be many ways to choose the invariant Lagrangians. A decoration is a special way to make this choice.

Definition 3.9. A *decoration* of ρ is a map

$$D\colon \pi_1^{per}(S) \to \mathrm{Lag}(2n,\mathbb{R})$$

satisfying the following properties:

(a) $D(g)$ is invariant under $\rho(g)$ for all $g \in \pi_1^{per}(S)$.

(b) If $g_1, g_2 \in \pi_1^{per}(S)$, $h \in \pi_1(S)$ such that $hg_1h^{-1} = g_2$, then

$$\rho(h)(D(g_1)) = D(g_2).$$

(c) For every $k \in \mathbb{Z} \smallsetminus \{0\}$ and for every $g \in \pi_1^{per}(S)$,

$$D(g) = D(g^k).$$

A *decorated representation* is a pair (ρ, D), where ρ is a representation and D a decoration of ρ.

Remark 3.10. By properties a), b), c) of decorations, for every puncture, one has to choose a Lagrangian for only one peripheral element going around the puncture. Then the Lagrangians associated to the other peripheral elements going around the same puncture are determined.

We denote by $\mathrm{Hom}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$ the set of all decorated representations. The action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Hom}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$ and on $\mathrm{Lag}(2n,\mathbb{R})$ induces an action on $\mathrm{Hom}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$. We will study the quotient:

Definition 3.11. The quotient space

$$\mathrm{Rep}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) := \mathrm{Hom}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))/\,\mathrm{Sp}(2n,\mathbb{R})$$

is called the *moduli space of decorated representations.* We denote by $[\rho, D]$ the class of (ρ, D) in the moduli space of decorated representation.

Remark 3.12. We have natural surjective maps

$$\begin{array}{ccc} \mathrm{Hom}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Hom}^P(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) \\ (\rho, D) & \mapsto & \rho \end{array}.$$

$$\begin{array}{ccc} \mathrm{Rep}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Rep}^P(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) \\ [\rho, D] & \mapsto & [\rho] \end{array}.$$

These maps are generically $2^nk : 1$-map, where k is the number of punctures.

3.3. **Framed representations.** For a unipotent representation there are many ways to choose these invariant framed Lagrangians. A framing is a special way to make this choice.

Definition 3.13. A *framing* of ρ is a map
$$v\colon \pi_1^{per}(S) \to \mathrm{Lag}^{fr}(2n,\mathbb{R})$$
satisfying the following properties:
(a) $v(g)$ is invariant by $\rho(g)$ for all $g \in \pi_1^{per}(S)$.
(b) If $g_1, g_2 \in \pi_1^{per}(S)$, $h \in \pi_1(S)$ such that $hg_1h^{-1} = g_2$, then
$$\rho(h)(v(g_1)) = v(g_2).$$
(c) For every $k \in \mathbb{Z} \smallsetminus \{0\}$ and for every $g \in \pi_1^{per}(S)$,
$$v(g) = v(g^k)$$

A *framed representation* is a pair (ρ, v), where v is a framing of ρ.

Remark 3.14. By properties a), b), c) of a framing, for every puncture, one has to choose a framed Lagrangian for only one peripheral element going around the puncture. Then the framed Lagrangians associated to the other peripheral elements going around the same puncture are determined.

We denote by $\mathrm{Hom}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ the set of all framed representations. The action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ and on $\mathrm{Lag}^{fr}(2n,\mathbb{R})$ induces an action on $\mathrm{Hom}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. We will study the quotient:

Definition 3.15. The quotient space
$$\mathrm{Rep}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) := \mathrm{Hom}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))/\,\mathrm{Sp}(2n,\mathbb{R})$$
is called *moduli space of framed representations*. We denote by $[\rho, v]$ the class of the framed representation (ρ, v).

Remark 3.16. We have natural surjective maps
$$\begin{array}{ccc} \mathrm{Hom}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Hom}^{U}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ (\rho, v) & \mapsto & \rho \end{array}.$$
$$\begin{array}{ccc} \mathrm{Rep}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Rep}^{U}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ [\rho, v] & \mapsto & [\rho] \end{array}.$$

Remark 3.17. Every framing v of ρ induces a decoration D_v by the rule
$$D_v(g) := \mathrm{Span}(v(g))$$
We have natural maps
$$\begin{array}{ccc} \mathrm{Hom}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Hom}^{d}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ (\rho, v) & \mapsto & (\rho, D_v) \end{array}.$$
$$\begin{array}{ccc} \mathrm{Rep}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Rep}^{d}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ [\rho, v] & \mapsto & [\rho, D_v] \end{array}.$$

3.4. **Transverse representations.** We now fix an ideal triangulation $\mathcal{T}$ of S.

Definition 3.18. We say that $(\rho, D) \in \mathrm{Hom}^d(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ is *transverse* with respect to $\mathcal{T}$ if the following condition holds: for every edge e of $\mathcal{T}$ connecting punctures p_i and p_j, for every point $b' \in Int(e)$ and for every curve γ connecting b and b', we require that the Lagrangians $D(\gamma * \alpha_i * \gamma^{-1})$ and $D(\gamma * \alpha_j * \gamma^{-1})$ are transverse, where the curves α_i and α_j are as in Figure 3.1.

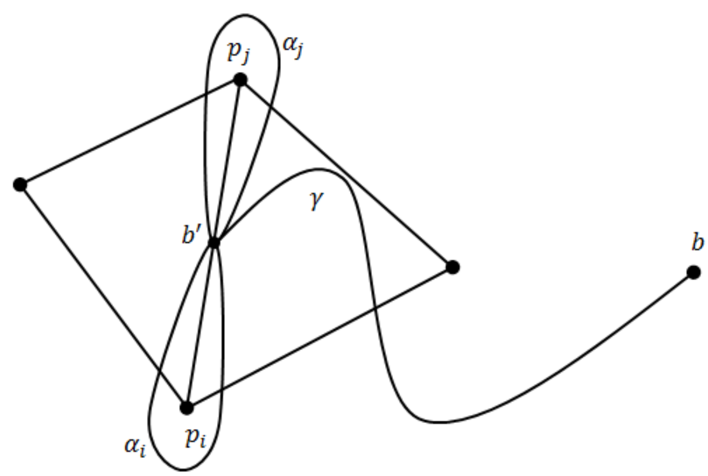

FIGURE 3.1.

We denote by $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ the set of all decorated representations which are transverse with respect to the triangulation $\mathcal{T}$.

Remark 3.19. The transversality property required in the previous definition does not depend on the choice of the path γ and the base point b. Moreover, this property is invariant under the action of $\mathrm{Sp}(2n,\mathbb{R})$, hence we can define the quotient:

$$\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) := \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))/\,\mathrm{Sp}(2n,\mathbb{R})$$

Remark 3.20. For each $\mathcal{T}$, the space $\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is an open dense subspace of $\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.

Definition 3.21. We denote by $\mathrm{Rep}^{fr}_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ the pre-image of $\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ under the map:

$$\begin{array}{ccc} \mathrm{Rep}^{fr}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) & \to & \mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \\ (\rho, v) & \mapsto & (\rho, D_v) \end{array}.$$

Let T be a triangle of $\mathcal{T}$ with boundary ∂T. Using the orientation of S, we can orient ∂T so that T is to the left from ∂T. This gives us a cyclic order on the vertices $\{p_1, p_2, p_3\}$ of T. We assume that (p_1, p_2, p_3) are in positive cyclic order.

Definition 3.22. Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$, and consider elements $g_1, g_2, g_3 \in \pi_1^{per}(S, b)$ that go around p_1, p_2, p_3 (see Figure 3.2). We can consider the Maslov index $\mu^T := \mu(D(g_1), D(g_2), D(g_3))$. Since μ is $\mathrm{Sp}(2n, \mathbb{R})$-invariant, μ^T is a well defined invariant of $[\rho, D]$ for each triangle T of $\mathcal{T}$. We call μ^T the *Maslov index of the positive oriented triangle* T for $[\rho, D]$.

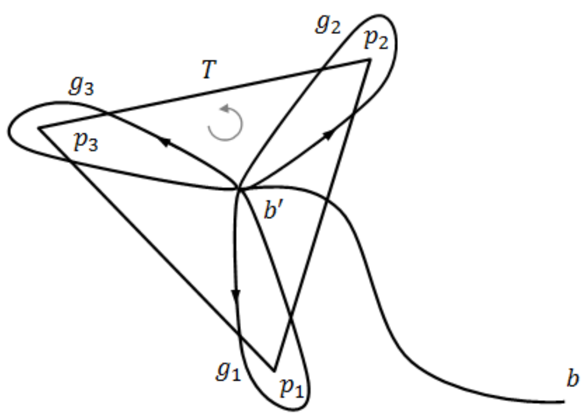

FIGURE 3.2.

3.5. **Toledo number and maximal representations.** An important invariant for representations $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is the Toledo number, here denoted by T_ρ, which was defined in [7] using bounded cohomology. It is a real number which satisfies the Milnor–Wood inequality:

$$-n|\chi(S)| \leq T_\rho \leq n|\chi(S)|.$$

Moreover, for all representations $[\rho] \in \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$, this invariant takes only integer values. The representations where this invariant achieves its maximum have particularly nice geometric properties, see [7].

Definition 3.23. A representation $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is called *maximal* if $T_\rho = n|\chi(S)|$.

We denote by $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the subspace of $\mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ consisting of all maximal representations. Similarly, we denote by $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the subspace of $\mathrm{Rep}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ of all decorated maximal representations, and by $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the subspace of all decorated maximal representations which are transverse with respect to a chosen triangulation $\mathcal{T}$. The following facts are proven in [7].

Proposition 3.24. *[7]*

(a) $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \subset \mathrm{Rep}^P(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. *In particular, the natural projection map*

$$\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \quad \to \quad \mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})).$$

is surjective.

(b) Maximal representations are transverse with respect to any ideal triangulation $\mathcal{T}$:

$$\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) = \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})).$$

(c) All maximal representations are reductive, hence the spaces $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ and $M^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ are Hausdorff (cfr. Remark 3.3).

Remark 3.25. A representation $[\rho] \in \mathrm{Rep}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is called *almost maximal* if $T_\rho > (n-1)|\chi(S)|$ (see [9]). The Remark 3.24 (c) holds also for the subsets of the moduli spaces consisting of all almost maximal representations.

We now show that the Toledo number of a decorated representation can be computed easily using an ideal triangulation. In the special case of a pair of pants the following proposition was proven in [21].

Proposition 3.26. *Let $\mathcal{T}$ be an ideal triangulation of S and $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. The Toledo number T_ρ of ρ can be computed from the following formula:*

$$T_\rho = \sum_{T \in \mathcal{T}} \mu^T$$

where μ^T is the Maslov index of the positive oriented triangle T for $[\rho, D]$.

Corollary 3.27. *The number $\sum_{T\in\mathcal{T}} \mu^T$ only depends on the representation. In particular it does not depend on the choice of decoration nor on the ideal triangulation.*

The fact that $\sum_{T\in\mathcal{T}} \mu^T$ does not depend on the triangulation can also be seen directly since every two triangulations are connected by a sequence of flips, and for a flip the statement follows from the cocycle relation of the Maslov index (see Remark 2.8).

As a corollary of the previous proposition, we can recognize decorated maximal representations using a triangulation:

Corollary 3.28. *Given a decorated representation ρ, and an ideal triangulation $\mathcal{T}$ of S, we have that ρ is maximal if and only if the Maslov index of each positively oriented triangle T in $\mathcal{T}$ is n.*

The proof of Proposition 3.26 will take the rest of this subsection. It will use, as tools, the Souriau index and the rotation number, whose properties we will briefly discuss.

Let $\tilde{G}$ be the universal covering of $G := \mathrm{Sp}(2n,\mathbb{R})$ and $\widetilde{\mathrm{Lag}}(2n,\mathbb{R})$ be the universal covering of $\mathrm{Lag}(2n,\mathbb{R})$. In [7] it is shown that $\tilde{G}$ acts on $\widetilde{\mathrm{Lag}}(2n,\mathbb{R})$

in a compatible way with respect to the action of G on $\mathrm{Lag}(2n,\mathbb{R})$, i.e. for all $\tilde{g}\in\tilde{G}$ and for all $\tilde{L}\in\widetilde{\mathrm{Lag}}(2n,\mathbb{R})$:

$$p(\tilde{g}.\tilde{L}) = p_G(\tilde{g}).p(\tilde{L})$$

where $p\colon \widetilde{\mathrm{Lag}}(2n,\mathbb{R})\to \mathrm{Lag}(2n,\mathbb{R})$, $p_G\colon \tilde{G}\to G$ are natural projections of coverings, and by . we denote the actions of corresponding groups.

The Souriau Index is a map

$$m\colon \widetilde{\mathrm{Lag}}(2n,\mathbb{R})\times\widetilde{\mathrm{Lag}}(2n,\mathbb{R})\to\mathbb{R}$$

which is $\tilde{G}$-invariant and satisfies the following relation: for each $\tilde{L}_1,\tilde{L}_2,\tilde{L}_3\in\widetilde{\mathrm{Lag}}(2n,\mathbb{R})$

$$m(\tilde{L}_1,\tilde{L}_2)+m(\tilde{L}_2,\tilde{L}_3)+m(\tilde{L}_3,\tilde{L}_1)=\mu(L_1,L_2,L_3)$$

where $L_i = p(\tilde{L}_i)$ for $i\in\{1,2,3\}$. See [8] and [21] for a precise definition.

We also need the rotation number $\widetilde{\mathrm{Rot}}\colon \tilde{G}\to\mathbb{R}$, a conjugation invariant function defined in [7] using the theory of bounded cohomology. We will need the following properties:

Lemma 3.29 ([21]). *Let $\tilde{g}\in\tilde{G}$, $\tilde{L}\in\widetilde{\mathrm{Lag}}(2n,\mathbb{R})$ and let $p(\tilde{L})\in\mathrm{Lag}(2n,\mathbb{R})$ be a fixed point of $p_G(\tilde{g})\in G$. Then*

$$\begin{array}{rccc}\widetilde{\mathrm{Rot}}\colon & \tilde{G} & \to & \mathbb{R}\\ & \tilde{g} & \mapsto & m(\tilde{g}\tilde{L},\tilde{L})\end{array}$$

Lemma 3.30 ([7, Thm. 12]). *Let $\rho\in\mathrm{Hom}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$ and*

$$\pi_1(S)=\langle a_1,b_1,\dots,a_g,b_g,c_1,\dots,c_k \mid c_1\dots{}_k\,[b_g,a_g]\dots[b_1,a_1]=1\rangle$$

be a presentation of $\pi_1(S)$. Let $\tilde{\rho}\in\mathrm{Hom}(\pi_1(S),\tilde{G})$ be a lift of ρ to the universal covering $\tilde{G}$ of $\mathrm{Sp}(2n,\mathbb{R})$. The Toledo number of ρ can be computed as:

$$T_\rho = -\sum_{i=1}^{k}\widetilde{\mathrm{Rot}}(\tilde{\rho}(c_i))$$

We are finally ready to present the

Proof of Proposition 3.26. First we fix a presentation of $\pi_1(S)$:

$$\pi_1(S)=\langle a_1,b_1,\dots,a_g,b_g,c_1,\dots,c_k \mid c_1\dots{}_k\,[b_g,a_g]\dots[b_1,a_1]=1\rangle$$

where g is the genus of S, k is the number of punctures. We choose a lift $\tilde{\rho}\colon\pi_1(S)\to\tilde{G}$. From $\tilde{\rho}$, we can compute T_ρ using Lemma 3.30.

We can assume that $\tilde{\rho}(c_i)$ have a fixed point z_i, $i\geq 2$ in $\widetilde{\mathrm{Lag}}(2n,\mathbb{R})$. So $\widetilde{\mathrm{Rot}}(\tilde{\rho}(c_i))=0$. This is possible since $\rho(c_2),\dots,\rho(c_k)$ have fixed points in $\mathrm{Lag}(2n,\mathbb{R})$. We also denote by y_0 a lift of a fixed point of $\rho(c_1)$.

We denote for all admissible i:

$$A_i := \rho(a_i),\ \tilde{A}_i := \tilde{\rho}(a_i),$$

$$B_i := \rho(b_i),\ \tilde{B}_i := \tilde{\rho}(b_i),$$

$$C_i := \rho(c_i),\ \tilde{C}_i := \tilde{\rho}(\tilde{c}_i).$$

By induction, we denote

$$y_i := \tilde{B}_i^{-1}\tilde{A}_i^{-1}\tilde{B}_i\tilde{A}_i y_{i-1}$$

for $i \in \{1, \dots, g\}$.

We consider a polygon model of S and the ideal triangulation as in Figure 3.3, where the vertices of the triangulation are decorated by lifted fixed points of the corresponding peripheral elements, and the edges are marked by letters and arrows corresponding to the generators of the fundamental group and gluing/cutting directions.

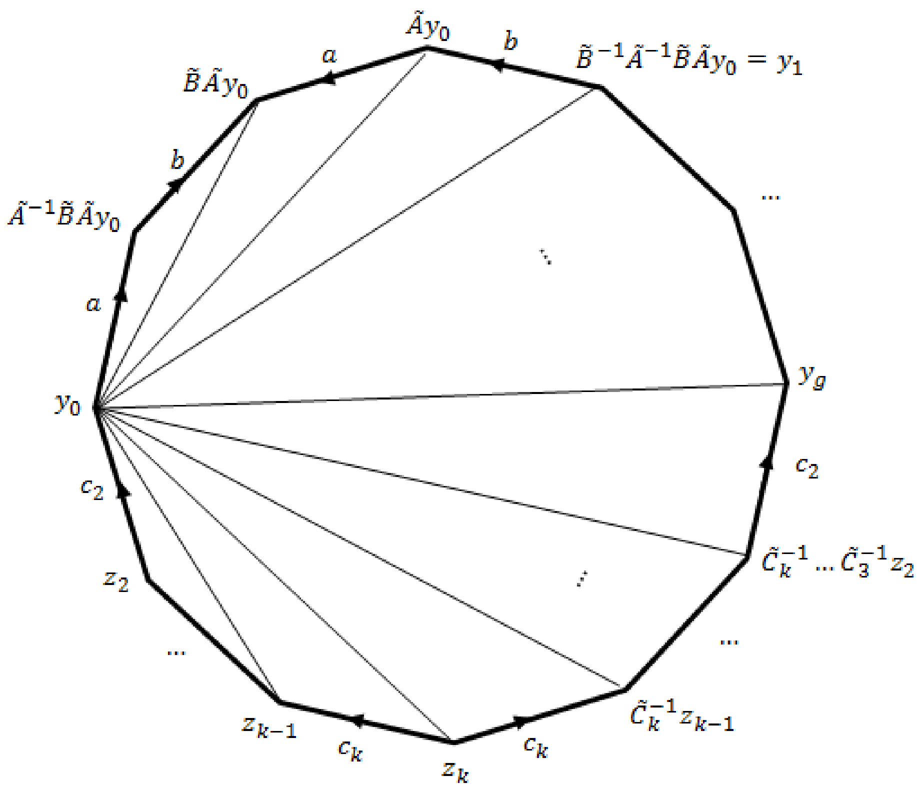

FIGURE 3.3.

To write the sum of Maslov indices, we use the Souriau index [21, 3.2]:

$$\sum_{T\in\mathcal{T}}\mu_\rho^T=\sum_{i=1}^g(m(y_{i-1},\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1})+m(\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1},\tilde B_i\tilde A_iy_{i-1})+$$
$$+m(\tilde B_i\tilde A_iy_{i-1},\tilde A_iy_{i-1})+m(\tilde A_iy_{i-1},\tilde B_i^{-1}\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1}))+$$
$$+m(y_g,\tilde C_k^{-1}\dots\tilde C_3^{-1}z_2)+$$
$$+\sum_{i=2}^{k-1}m(\tilde C_k^{-1}\dots\tilde C_{i+1}^{-1}z_i,\tilde C_k^{-1}\dots\tilde C_{i+2}^{-1}z_{i+1})+\sum_{i=2}^{k-1}m(z_{i+1},z_i)+m(z_2,y_0).$$

Using the $\tilde G$-invariance of the Souriau index and its anti-symmetry we can see that

$$m(\tilde B_i\tilde A_iy_{i-1},\tilde A_iy_{i-1})=m(\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1},y_{i-1})=-m(y_{i-1},\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1})$$
$$m(\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1},\tilde B_i\tilde A_iy_{i-1})=m(\tilde B_i^{-1}\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1},\tilde A_iy_{i-1})=$$
$$=-m(\tilde A_iy_{i-1},\tilde B_i^{-1}\tilde A_i^{-1}\tilde B_i\tilde A_iy_{i-1}).$$

Therefore, the first sum is equal to zero. Moreover,

$$m(\tilde C_k^{-1}\dots\tilde C_{i+1}^{-1}z_i,\tilde C_k^{-1}\dots\tilde C_{i+2}^{-1}z_{i+1})=m(\tilde C_{i+1}^{-1}z_i,z_{i+1})=$$
$$=m(z_i,\tilde C_{i+1}z_{i+1})=m(z_i,z_{i+1}).$$

Therefore, the second sum is equal to minus the third sum. So we get:

$$\sum_{T\in\mathcal{T}}\mu^T=m(y_g,\tilde C_k^{-1}\dots\tilde C_3^{-1}z_2)+m(z_2,y_0)=$$
$$=m(y_g,\tilde C_k^{-1}\dots\tilde C_3^{-1}\tilde C_2^{-1}z_2)+m(z_2,y_0)=$$
$$=m(\tilde C_2\tilde C_3\dots\tilde C_ky_g,z_2)+m(z_2,y_0)=m(\tilde C_1^{-1}y_0,z_2)+m(z_2,y_0)=$$
$$=m(\tilde C_1^{-1}y_0,y_0)=\widetilde{\mathrm{Rot}}(\tilde C_1^{-1})=-\widetilde{\mathrm{Rot}}(\tilde C_1)=T_\rho.\quad\square$$

4. $\mathcal{X}$-coordinates for maximal representations

In this section we introduce positive $\mathcal{X}$-coordinates. They will give a parametrization of the space of maximal representations: we restrict our attention here to this special case because the definition is significantly simpler than in the general case. The definition of general $\mathcal{X}$-coordinates for decorated representations that are not necessarily maximal will be given in Section 6.

4.1. **Ideal triangulations of surfaces.** Let $\mathcal{T}$ be an ideal triangulation of a punctured surface $S=\bar S\smallsetminus P$ where $P=\{p_1,\dots,p_k\}$ is the set of punctures. We consider $\mathcal{T}$ as a graph $\mathcal{T}=(P,E)$ embedded in $\bar S$ so that the complement of $\mathcal{T}$ in $\bar S$ is a disjoint union of triangles which we call faces or triangles of the triangulation $\mathcal{T}$. We denote by F the set of all faces of $\mathcal{T}$.

The $\mathcal{X}$-coordinates will in general consist face invariants, edge invariants, and angle invariants. The face coordinates essentially come form the Maslov index, so they take values in a discrete set, and for positive $\mathcal{X}$-coordinates, they are all constant equal to n, so that we can suppress them. For the angle coordinates it is important to introduce the angles of the triangulation, which is what we do know.

For each edge $e \in E$ there are up to homotopy two parametrizations $\vec{e}\colon [0,1] \to e$ and $\vec{e}^{\,-1}\colon [0,1] \to e$, where $\vec{e}^{\,-1}(t) = \vec{e}(1-t)$. The restrictions $\vec{e}, \vec{e}^{\,-1}\colon (0,1) \to e \setminus P$ are bijective. The choice of $\vec{e}$ for $e \in E$ is called an orientation of the edge $e \in E$. We denote by E_{or} the set of all oriented edges of $\mathcal{T}$.

The orientation of S defines maps:

$$r\colon E_{or} \to F$$

$$l\colon E_{or} \to F$$

which associate to an oriented edge $\vec{e}$ the unique face whose closure contains this edge and which lies to the right (resp. to the left) of $\vec{e}$.

Definition 4.1. An ideal triangulation $\mathcal{T}$ together with a chosen orientation for every edge is called an *oriented ideal triangulation.*

Definition 4.2 (Positive and negative angles)**.** The triple $(\vec{e}_1, \vec{e}_2, f) \in E_{or}^2 \times F$ is called a *positive angle* of the triangulation $\mathcal{T}$ if

- $\vec{e}_1(1) = \vec{e}_2(0) \subseteq P \cap \bar{f}$,
- $l(\vec{e}_1) = l(\vec{e}_2) = f$.

Similarly, the triple $(\vec{e}_1, \vec{e}_2, f) \in E_{or}^2 \times F$ is called a *negative angle* of the triangulation $\mathcal{T}$ if

- $\vec{e}_1(1) = \vec{e}_2(0) \subseteq P \cap \bar{f}$,
- $r(\vec{e}_1) = r(\vec{e}_2) = f$.

We denote by W^+ (resp. W^-) the set of all positive (resp. negative) angles of $\mathcal{T}$, and by W the set of all angles of $\mathcal{T}$, i.e. $W = W^+ \cup W^-$.

For each angle $w = (\vec{e}_1, \vec{e}_2, f)$ the opposite angle is defined as:

$$w^{-1} = (\vec{e}_2^{\,-1}, \vec{e}_1^{\,-1}, f) \in W.$$

Obviously, the opposite angle of a positive angle is negative and vice versa.

Definition 4.3 (Positive triple)**.** We call a triple of different positive angles (w_1, w_2, w_3) *positive* if

$$w_1 = (\vec{e}_1, \vec{e}_2, f),\ w_2 = (\vec{e}_2, \vec{e}_3, f),\ w_3 = (\vec{e}_3, \vec{e}_1, f)$$

for $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in E_{or}$.

Obviously, the positivity of a triple of positive angles is invariant under cyclic permutations.

For simplicity we will draw orientation of angles using arrows as on Figure 4.1

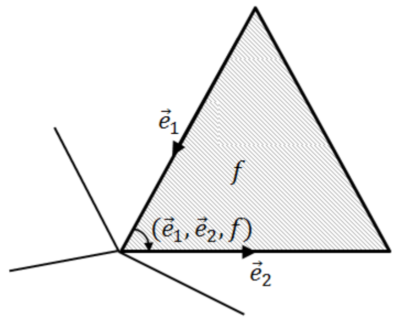

FIGURE 4.1.

4.2. **Positive $\mathcal{X}$-coordinates.** Let S be a surface with an oriented ideal triangulation $\mathcal{T}$. We use the notation introduced in Section 4.1.

Definition 4.4 (Positive $\mathcal{X}$-coordinates)**.** A *system of positive $\mathcal{X}$-coordinates* of rank n on $(S,\mathcal{T})$ is a map

$$x\colon E \sqcup W^+ \to \mathbb{R}^n_{>0} \sqcup O(n)$$

such that

- the *edge invariant*$x(e)$ for an edge $e \in E$ is an n-tuple of positive real numbers $x(e) = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{>0}$ with $\lambda_i \geq \lambda_{i+1}$;
- the *angle invariant* $x(w)$ for a positive angle $w \in W^+$ is an orthogonal matrix $x(w) \in \mathrm{O}(n)$. The angle coordinates are subject to the following relation: for each positive triple of positive angles (w_1, w_2, w_3) we require

$$x(w_3)x(w_2)x(w_1) = \mathrm{Id}\,.$$

We denote by $\mathcal{X}^+(S,\mathcal{T},n)$ the set of all positive systems of $\mathcal{X}$-coordinates of rank n on $(S,\mathcal{T})$.

As a convenient notation, if $x \in \mathcal{X}^+(S,\mathcal{T},n)$ is a system of $\mathcal{X}$-coordinates and $w \in W^-$ is a negative angle, we will write $x(w) = x(w^{-1})^{-1}$.

Given a system of positive $\mathcal{X}$-coordinates, we can construct a decorated transverse homomorphism of the fundamental group $\pi_1(S,b)$ for an appropriately chosen $b \in S$. We describe this procedure in two steps, first constructing the homomorphism and then the decoration.

For this we lift the triangulation $\mathcal{T}$ of S to a triangulation $\tilde{\mathcal{T}} = (\tilde{P}, \tilde{E})$ of the universal covering $\tilde{S}$ of S.

We define a graph Γ on the surface in the following way: in every triangle we choose three points close to the three edges, these points will be the vertices of the graph. The edges of Γ are segments connecting the three points in one triangle and segments connecting the two points in neighboring triangles that are close to the same edge of the triangulation (see Figure 4.2).

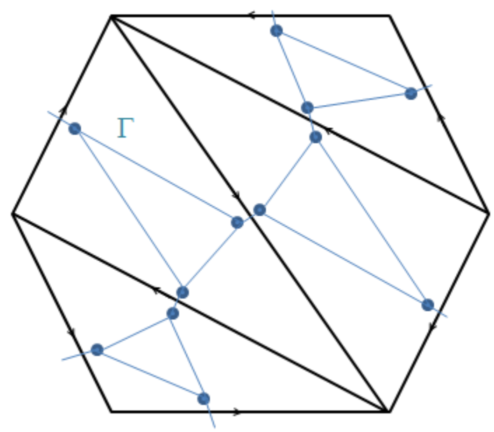

FIGURE 4.2.

We assume that the base point b coincide with one of vertices of Γ. Now, every element $\alpha \in \pi_1(S,b)$ has a representative which is a closed simplicial path in the graph Γ. We can write α as composition of paths

$$\alpha = \alpha_k \circ \cdots \circ \alpha_1,$$

where every α_i is a path along one edge of Γ.

To define the representation $\rho = \mathrm{rep}^+(x)$, we will associate to every α the matrix

$$\rho(\alpha) = A_k \cdots A_1.$$

We introduce the following notation, if $x(r)$ is an edge invariant, i.e. it a an n-tuple of positive real numbers $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{>0}$ with $\lambda_i \geq \lambda_{i+1}$, then $\mathrm{diag}(x(r))$ denotes the diagonal matrix whose ith-entry is λ_i.

Then A_i is defined as follows:

- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from *the right to the left* assuming that the edge $\vec{r}$ is oriented upwards, we have
$$A_i := \begin{pmatrix} 0 & -\sqrt{\mathrm{diag}(x(r))} \\ \sqrt{\mathrm{diag}(x(r))}^{\,-1} & 0 \end{pmatrix}$$
where $\sqrt{\mathrm{diag}(x(r))}$ is a coordinatewise positive square root.
- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from *the left to the right* assuming that the edge $\vec{r}$ is oriented upwards, we have
$$A_i := -\begin{pmatrix} 0 & -\sqrt{\mathrm{diag}(x(r))} \\ \sqrt{\mathrm{diag}(x(r))}^{\,-1} & 0 \end{pmatrix}$$
where $\sqrt{\mathrm{diag}(x(r))}$ is a coordinatewise positive square root.

- If α_i is along an edge of Γ that follows the angle w of the triangulation, consider the matrices

$$\hat{U} := \begin{pmatrix} x(w)^T & 0 \\ 0 & x(w)^T \end{pmatrix},$$

$$T_r = \begin{pmatrix} -\operatorname{Id} & \operatorname{Id} \\ -\operatorname{Id} & 0 \end{pmatrix}, \quad T_l = (T_r)^{-1}.$$

  We have $A_i = T_r\hat{U}$ (resp. $A_i = T_l\hat{U}$) if when going from α_{i-1} to α_i we are turning to the right (resp. to the left). Notice that, T_r and $\hat{U}$ commute: $T_r\hat{U} = \hat{U}T_r$, $T_l\hat{U} = \hat{U}T_l$.

All the matrices A_i are symplectic, so $\rho(\alpha) \in \operatorname{Sp}(2n, \mathbb{R})$. It is easy to check that this matrix only depends on the homotopy class of α, and that the map is a group homomorphism. In this way we constructed a representation $\rho \in \operatorname{Hom}(\pi_1(S, b), \operatorname{Sp}(2n, \mathbb{R}))$.

We now construct a decoration D for this representation. First, consider the case of a puncture that is a vertex of an edge of $\mathcal{T}$ which is close to the basepoint b. A simple peripheral element of $\pi_1(S, b)$ around this puncture can be represented by a circle c going around this puncture. Then going around c we always are turning either to the right or to the left. Therefore, either $L_{\mathbf{e}} = \operatorname{Span}(\mathbf{e})$ or $L_{\mathbf{f}} = \operatorname{Span}(\mathbf{f})$ is preserved by $\rho(c)$, where $(\mathbf{e}, \mathbf{f})$ is the standard symplectic basis of $(\mathbb{R}^{2n}, \omega)$ (see Figure 4.3).

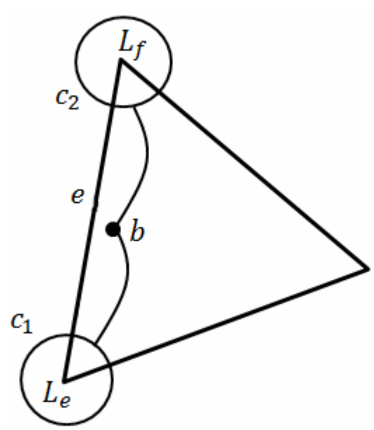

FIGURE 4.3.

Now we extend this definition to general punctures. First, we note that if α is any path in the graph Γ, we write $\alpha = \alpha_k \circ \cdots \circ \alpha_1$, where every α_i is a path along one edge of Γ. The definition of the matrix $\rho(\alpha)$ given above can be applied also to this path α, even if it is not closed.

Finally, for each simple peripheral curve γ around some puncture p with start- and endpoint b, we can take a point b' which lies in a triangle adjacent to p. Then we can decompose γ up to homotopy into a path α from b to b', circle c around p and the inverse path α^{-1} from b' to b. The representation ρ associates to this element the matrix

$$\rho(\gamma) = \rho(\alpha^{-1})\rho(c)\rho(\alpha)$$

We have already seen how to construct a Lagrangian L preserved by the matrix $\rho(c)$, we can then associate to γ the matrix $D(\gamma) := \rho(\alpha^{-1})L$.

For each non-simple peripheral curve which is a power of some simple one, we define a decoration of non-simple peripheral curve to be the decoration of the corresponding simple curve. All other non-simple curves are of the form $\gamma = \beta^{-1}\alpha^n\beta$, where α is simple closed curve, β is some closed curve. So we define $D(\gamma) := \rho(\beta).D(\alpha)$.

In this way, starting from a system of X-coordinates x, we defined an element $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$. We define $\mathrm{rep}^+(x) := (\rho, D)$.

4.3. **Properties of the map** rep^+**.** We now describe properties of the map $\mathrm{rep}^+ : \mathcal{X}^+(S,\mathcal{T},n) \to \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$.

For this we introduce the notion of coordinates that are admissible with respect to a decorated representation $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$. Note that we can lift the decoration D to a map $\tilde{D}\colon \tilde{P} \to \mathrm{Lag}(2n,\mathbb{R})$.

Definition 4.5. $x \in \mathcal{X}^+(S,\mathcal{T},n)$ is called *admissible* for a maximal representation $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$ if

- for each edge $e \in \tilde{E}$ on the boundary of the triangles $T = (t_1, t_3, t_2)$ and $T' = (t_2, t_4, t_1)$ of $\tilde{\mathcal{T}}$, the cross ratio $[\tilde{D}(t_1), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$ is conjugated to $-\operatorname{diag}(x(e))$;
- for each pentagon in $\tilde{\mathcal{T}}$ as in Figure 4.4, the orthogonal matrix $x(w)$ belongs to the double coset $[\tilde{D}(t_1), \tilde{D}(t_5), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$.

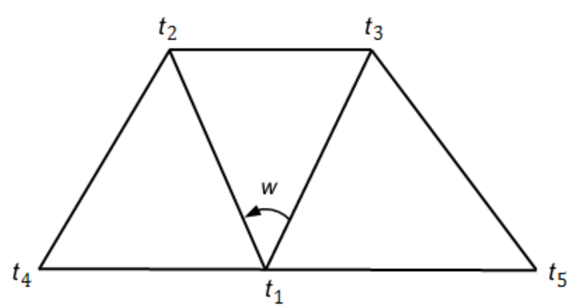

FIGURE 4.4.

Remark 4.6. This definition is independent on the choice of $(\rho, D) \in [\rho, D]$ and of the lift $\tilde{D}$ of D.

Proposition 4.7. *For every $x \in X^+(S,\mathcal{T},n)$, the image $\mathrm{rep}^+(x)$ is a decorated maximal representation , and x is admissible for the representation $\mathrm{rep}^+(x)$.*

Proof. A direct calculation in one triangle shows that for the decoration constructed above each positive oriented triangle has maximal Maslov index. Similarly, a direct calculations in a quadrilateral and in a pentagon show admissibility of x for $\mathrm{rep}(x)$. □

We denote by $[\mathrm{rep}^+](x)$ the conjugacy class of $\mathrm{rep}^+(x)$. We just constructed a map

$$[\mathrm{rep}^+]\colon \mathcal{X}^+(S,\mathcal{T},n) \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S,b),\mathrm{Sp}(2n,\mathbb{R})).$$

This map is surjective (see Corollary 4.13) but it is not injective: sometimes changing the angle coordinates, the image representation stays the same. We describe this ambiguity explicitly.

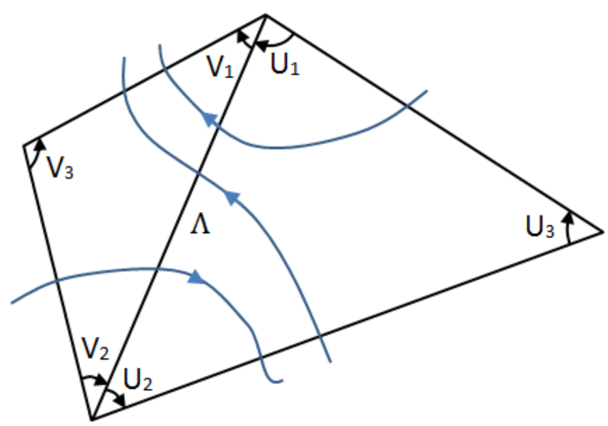

FIGURE 4.5.

Proposition 4.8. *Let $x \in \mathcal{X}^+(S,\mathcal{T},n)$. Consider two triangles adjacent by an edge e. Let $x(e) = \Lambda$ and consider the angle coordinates be defined as in Figure 4.5. Let us change angle coordinates in the following way:*

$$U_1' = WU_1,\ V_1' = V_1W'^{-1},$$

$$U_2' = U_2W^{-1},\ V_2' = W'V_2.$$

We denote by x' the changed coordinates. Then $[\mathrm{rep}^+](x) = [\mathrm{rep}^+](x')$ if and only if

$$W \in \mathrm{O}(n) \cap \mathrm{O}(\mathrm{diag}(\Lambda)),$$

$$W' := D^{-1}W^TD,$$

$$D := \sqrt{\mathrm{diag}\,\Lambda}.$$

Moreover, if $[\mathrm{rep}^+](x) = [\mathrm{rep}^+](x')$ for some $x, x' \in \mathcal{X}^+(S,\mathcal{T},n)$ then $x(e) = x'(e)$ for all edges e and there exists a finite sequence of changing of angle coordinates defined by formulas above which puts $x(w)$ to $x'(w)$ for all angles w.

Remark 4.9. The ambiguity in a choice of angle coordinated around an edge e depends on how generic the tuple $x(e) =: \Lambda$ is. Let $\lambda_1 > \cdots > \lambda_k$ are different entries of Λ with multiplicities $l_1, \dots, l_k$, then $W \in \mathrm{O}(l_1) \times \cdots \times \mathrm{O}(l_k) \leq \mathrm{O}(n)$ (diagonally embedded). In particular, for generic Λ with all entries different, $W \in \mathbb{Z}_2^n$. On the other hand, if $\Lambda = (\lambda, \dots, \lambda)$ for some $\lambda > 0$, then $W \in \mathrm{O}(n)$.

Proposition 4.8 will be proven in Section 6 where we treat general $\mathcal{X}$-coordinates.

4.4. **The set of positive $\mathcal{X}$-coordinates associated to a representation.** So far we only constructed a decorated maximal representation given a system of positive $\mathcal{X}$-coordinates. Now we describe how, given an ideal triangulation, we can associate a system of positive $\mathcal{X}$-coordinates to a decorated maximal representation $[(\rho, D)]$ so that $[\mathrm{rep}^+(x)] = [(\rho, D)]$. The basic idea is clear, we want a system of coordinates that is admissible for $[(\rho, D)]$ - so essentially for each edge e of the triangulation there are two adjacent triangles, whose vertices are decorated by four Lagrangian subspaces L_1, L_2, L_3, L_4, and the edge invariant $x(e)$ is the ordered set of eigenvalues of the cross ratio map $[L_1, L_2, L_3, L_4] : L_1 \to L_1$, and for every angle, we have a decoration by five Lagrangians, and the angle coordinate is the angle $[L_1, L_2, L_3, L_4, L_5]$, see Figure 2.1. However one has to be a bit careful when making the precise definitions, because we do not only want the the system of coordinates is admissible with respect to $[(\rho, D)]$, but that moreover that $[\mathrm{rep}^+(x)] = [(\rho, D)]$. And in general there are admissible system of $\mathcal{X}$-coordinates $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ for $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ such that $[\mathrm{rep}^+](x) \neq [\rho, D]$.

So we take an ideal triangulation $\mathcal{T}$ of S and choose $b_0 \in S$. Let $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S, b_0), \mathrm{Sp}(2n, \mathbb{R}))$ be a decorated maximal representation.

We lift the oriented triangulation $\mathcal{T}$ of S to the oriented triangulation $\tilde{\mathcal{T}}$ of the universal covering $\tilde{S}$. We also fix a lift $b \in \tilde{S}$ of $b_0 \in S$. Punctures are lifted to visual boundary points of $\tilde{S}$ (after choice of some Riemannian metric of finite area). Using the decoration D, each boundary point can be decorated by a Lagrangian in a unique way. This decoration is $\pi_1(S, b_0)$-equivariant.

We consider the graph Γ associated to this triangulation as in Section 4.2, see Figure 4.2. We can assume that Γ is invariant under the action of $\pi_1(S, b_0)$ on $\tilde{S}$. First, we associate a symplectic basis to each vertex of Γ and a tuple $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$ to each edge of lifted triangulation $\mathcal{T}$.

For each vertex b of Γ there is the unique edge r close to which this vertex lies and unique triangle T in which b lies. We take an orientation of the edge $\vec{r}$ such that the vertex b lies to the right from $\vec{r}$. We consider the triangle, which is adjacent to T across the edge r. Thus we have a quadrilateral decorated by Lagrangians (L_1, L_3, L_2, L_4). Since the representation is maximal, the bilinear form $\beta_3 := [L_1, L_2, L_3] \colon L_1 \to L_1^*$ is well defined and positive definite, and the cross ratio map $F := [L_1, L_3, L_2, L_4]\colon L_1 \to L_1$ is well defined and symmetric with respect to β_3 with positive eigenvalues.

We say that the four tuple (L_1, L_2, L_3, L_4) is in standard position with respect to a symplectic basis $(\mathbf{e}, \mathbf{f})$ if $L_1 = L_{\mathbf{e}}$, and $L_2 = L_{\mathbf{f}}$, $[L_3]_{\mathbf{e},\mathbf{f}} = \mathrm{Id}$, and $[L_4]_{\mathbf{e},\mathbf{f}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where $[F]_{\mathbf{e}} = -\operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

We then define the edge invariant $x(r) = x(\vec{r}) = (\lambda_1, \dots, \lambda_n)$ and associate the symplectic bases $B(b) = (\mathbf{e}, \mathbf{f})$ to the vertex b of Γ.

Because the oriented edge $\vec{r}$ defines the point b uniquely, sometimes we will say that the basis $B(b)$ is associated to the oriented edge $\vec{r}$ and write $B(\vec{r})$.

By construction, the map x for oriented edges is $\pi_1(S, b_0)$-invariant, therefore, x is well-defined for oriented edges of triangulation $\mathcal{T}$ of S. Moreover, the easy calculation shows that $x(\vec{r}) = x(\vec{r}^{\,-1})$, therefore $x(r)$ is well defined and does not depend on the choice of orientation. We have to take care of two other issues:

(1) For each oriented edge $\vec{r}$ of triangulation there are two vertices b_1, b_2 of Γ lying close to $\vec{r}$. In general, there are many possibilities to define $B(b_2)$ if $B(b_1)$ is fixed. We fix one of them, which is consistent with the construction of the map rep^+, namely with the matrix associated to the crossing of an edge. Assume $\vec{r}$ is oriented upwards, b_1 lies to the right from $\vec{r}$ and b_2 lies to the left. Let $B(b_1) =: (\mathbf{e}, \mathbf{f})$ then $B(b_2) := (-\mathbf{f}\sqrt{\operatorname{diag}(\lambda_1, \dots, \lambda_n)}, \mathbf{e}\sqrt{\operatorname{diag}(\lambda_1, \dots, \lambda_n)}^{\,-1})$ where $x(r) = (\lambda_1, \dots, \lambda_n)$.

(2) The choice of bases B is in general not unique. But it can always be chosen in a ρ-equivariant way with respect to the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases because the lifted decoration by Lagrangians is ρ-equivariant. We will always assume that B is ρ-equivariant.

To define the angle coordinate, we consider a pentagon decorated by Lagrangians as on the Figure 4.6. To each oriented diagonal $\vec{r}_0$ and $\vec{r}_1$ of this pentagon are associated bases $B(\vec{r}_0) =: (\mathbf{e_0}, \mathbf{f_0})$ of (L_1, L_2) and $B(\vec{r}_1) =: (\mathbf{e_1}, \mathbf{f_1})$ of (L_3, L_1). So we can define the angle invariant $x(w)$ to be $x(w) := [L_1, L_5, L_3, L_2, L_4]_{\mathbf{e_0},\mathbf{f_1}}$.

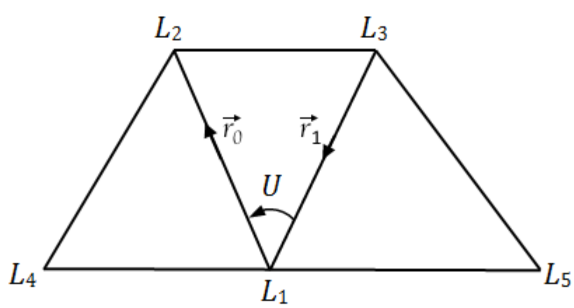

FIGURE 4.6.

Remark 4.10. Since the map B is ρ-equivariant, the map x for angles is $\pi_1(S, b_0)$-invariant. Therefore, x is well-defined for all oriented angles of the triangulation $\mathcal{T}$ of S.

Remark 4.11. Ordered tuple $x(r) = (\lambda_1, \ldots, \lambda_n)$ for each edge r is uniquely defined. In contrast, the matrices U for each angle are in general not uniquely defined by the representation ρ. To define U, we have chosen a map B fixing a symplectic basis for each oriented edge which is not unique in general.

Lemma 4.12. *Let* $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$. *Consider* $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ *constructed from* $[\rho, D]$ *as above. Then* $[\mathrm{rep}^+](x) = [\rho, D]$.

Proof. Notice, the bases on vertices of Γ were chosen in compatible way with the construction of the map rep^+, i.e. let b_1, b_2 be vertices of Γ connected by an edge r. To r the matrix E is associated as in the previous section (going along an angle or crossing an edge of triangulation). Then E maps the basis $B(b_1)$ to $B(b_2)$.

Therefore, by induction, for every loop α based in b, $\mathrm{rep}^+(\alpha)(B(b)) = B([\alpha]b)$, where by $[\alpha]b$ we understand the action of $[\alpha] \in \pi_1(S, b)$ on vertices of $\Gamma \subseteq \tilde{S}$. But the choice of B is ρ-equivariant, i.e. $\mathrm{rep}^+(\alpha)(B(b)) = B([\alpha]b) = \rho(\alpha)B(b)$. But the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases is exact, therefore, $\mathrm{rep}^+(\alpha) = \rho(\alpha)$ for all $[\alpha] \in \pi_1(S, b)$, where $\rho(\alpha)$ is written as a matrix with with respect to the basis $B(b)$. □

Corollary 4.13. *The map* $[\mathrm{rep}^+]$ *is surjective.*

4.5. **Change of coordinates.** The constructions of positive $\mathcal{X}$-coordinates depends on a choice of ideal triangulation $\mathcal{T}$ of S, however the representation $[\mathrm{rep}^+(x)]$ is independent of the triangulation. If we choose a different ideal triangulation $\mathcal{T}'$ we get a different set of positive $\mathcal{X}$-coordinates. In the work of Fock and Goncharov, it was essential that the coordinate changes going from one triangulation to another are given by positive rational functions, because these implies that the set of positive representations is independent of the triangulation used to define it. Here we know here a priori that the image of $[\mathrm{rep}^+]$ is independent of the triangulation, because it is the set of maximal representations, which can be defined without reference to any triangulation. It is of interest of interest to understand the coordinate changes.

Since every ideal triangulation $\mathcal{T}'$ can be obtained from any other ideal triangulation $\mathcal{T}$ by a sequence of flips, i.e. changing the triangulation just by taking a quadrilateral and exchanging one diagonal for the other one, the coordinate change of a flip is the central ingredient.

In the case of positive $\mathcal{X}$-coordinates it is quite difficult to write explicit formulas for this coordinate change. In particular the angle coordinates are given rather implicitly. However in the case of "scalar" edge invariants. Let $x(r) = l\,\mathrm{Id}$ then

$$\tilde{U}_1 = U_1U_2,\ \tilde{V}_1 = V_2V_1$$

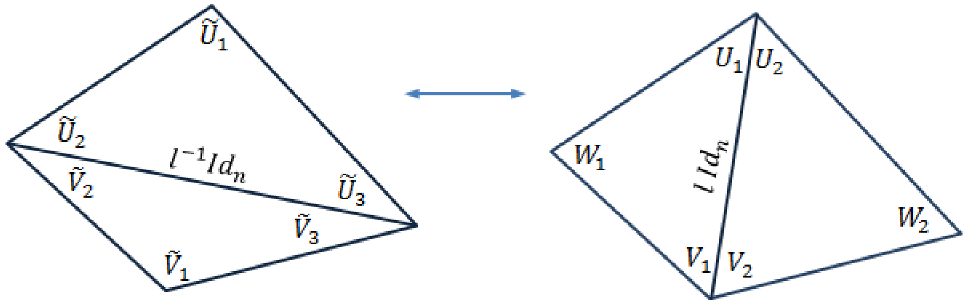

FIGURE 4.7. Flip along “scalar” edge

$$W_1 = \tilde{V}_2\tilde{U}_2,\ W_2 = \tilde{U}_3\tilde{V}_3$$

(triangles and angles are oriented counterclockwise).

In section 8.4 we give a nice formula for the flip after applying a local change of coordinates such that in the quadrilateral where we perform the flip, every edge is labeled by a symmetric $n \times n$ matrix. With this local change of coordinates the formulas for the flip look like noncommutative analogues of the formula for the flip for representations into $\mathrm{SL}(2,\mathbb{R})$.

4.6. **Comparison with Fock–Goncharov coordinates.** In this section we show that a maximal representation is not always positive in terms of Fock–Goncharov coordinates [10]. To do this, we take a positive 4-tuple of Lagrangians and show that it does not have always positive Fock–Goncharov coordinates.

To do this, first, we fix some symplectic basis $(\mathbf{e},\mathbf{f}) = (e_1,e_2,f_1,f_2)$ on $(\mathbb{R}^4,\omega)$ and consider the following four Lagrangians: $L_1 := L_{\mathbf{e}}$, $L_2 := L_{\mathbf{f}}$, $L_3 := L_{\mathbf{e},\mathbf{f}}(\mathrm{Id})$, $L_4 := L_{\mathbf{e},\mathbf{f}}(-\mathrm{Id})$. This 4-tuple has as $\mathcal{X}$-coordinate $(1,\dots,1)$.

Since the Fock–Goncharov coordinates are defined for decorations by full flags, we have to choose a line in each Lagrangian. We choose:

$$l_1 = \langle e_1 + \theta e_2\rangle \le L_1$$

$$l_2 = \langle f_1 + \lambda f_2\rangle \le L_2$$

$$l_3 = \langle e_1 + f_1 + \mu(e_2+f_2)\rangle \le L_3$$

$$l_4 = \langle e_1 - f_1 + \nu(e_2-f_2)\rangle \le L_4$$

where $\theta,\lambda,\mu,\nu \in \mathbb{R}$ some constants. Then the corresponding full flag for each $i \in \{1,2,3,4\}$ is $(l_i, L_i, l_i^{\perp})$, where $l_i^{\perp} = \{v \in \mathbb{R}^4 \mid \omega(l_i,v) = 0\}$.

So we get the following coordinates:

$$D_1 = -\frac{(\mu\theta+1)(\lambda-\nu)}{(\nu\theta+1)(\lambda-\mu)} \quad D_2 = \frac{(\lambda-\mu)(\theta-\nu)}{(\theta-\mu)(\lambda-\nu)} \quad D_3 = -\frac{(\nu\lambda+1)(\theta-\mu)}{(\mu\lambda+1)(\theta-\nu)}$$

$$T_1 = -\frac{(\mu\theta+1)(\theta-\lambda)}{(\lambda\theta+1)(\theta-\mu)} \quad T_2 = -\frac{(\lambda\theta+1)(\lambda-\mu)}{(\lambda\mu+1)(\lambda-\theta)} \quad T_3 = -\frac{(\lambda\mu+1)(\mu-\theta)}{(\theta\mu+1)(\mu-\lambda)}$$

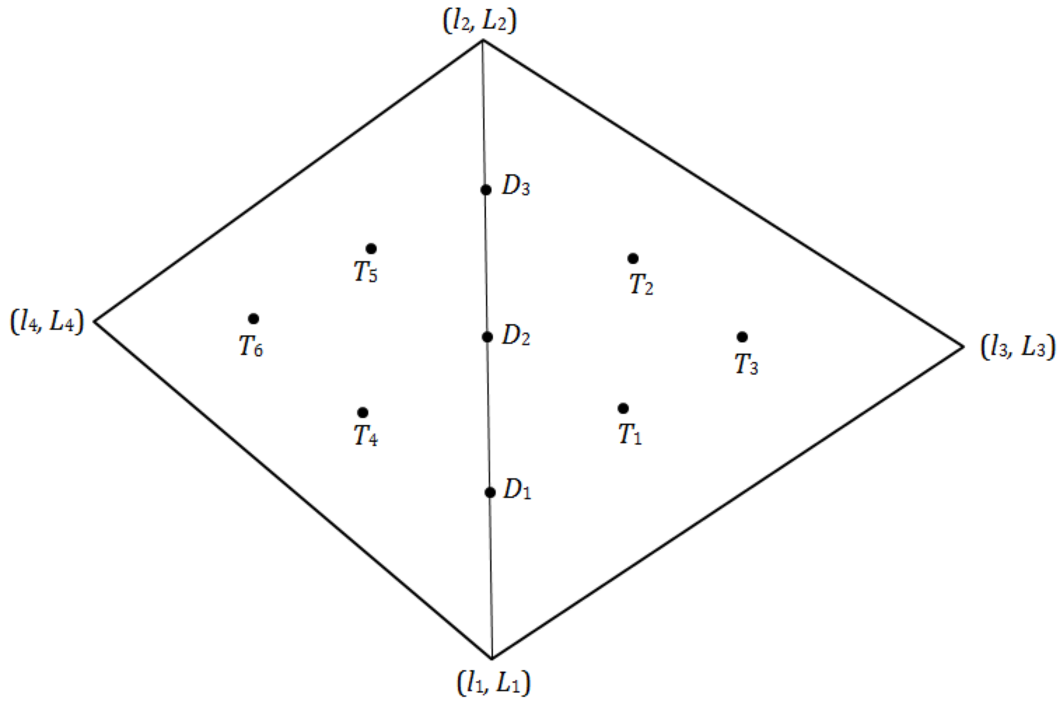

FIGURE 4.8.

$$T_4 = -\frac{(\lambda\theta+1)(\theta-\nu)}{(\nu\theta+1)(\theta-\lambda)} \quad T_5 = -\frac{(\nu\lambda+1)(\lambda-\theta)}{(\theta\lambda+1)(\lambda-\nu)} \quad T_6 = -\frac{(\theta\nu+1)(\nu-\lambda)}{(\lambda\nu+1)(\nu-\theta)}$$

We are going to show that all these coordinates can not be all positive for fixed $\theta, \lambda, \mu, \nu \in \mathbb{R}$. Assume first:

$$\frac{\theta-\lambda}{\lambda\theta+1} > 0$$

Since $T_2 > 0$, we get

$$\frac{\lambda-\mu}{\lambda\mu+1} > 0$$

Since $T_5 > 0$, we get

$$\frac{\lambda-\nu}{\nu\lambda+1} > 0$$

Therefore,

$$D_2 D_3 = -\frac{(\lambda-\mu)(\theta-\nu)}{(\theta-\mu)(\lambda-\nu)} \frac{(\nu\lambda+1)(\theta-\mu)}{(\mu\lambda+1)(\theta-\nu)} = -\frac{(\nu\lambda+1)(\lambda-\mu)}{(\mu\lambda+1)(\lambda-\nu)} < 0$$

and D_2 and D_3 cannot be positive at the same time.

If we assume

$$\frac{\theta-\lambda}{\lambda\theta+1} < 0$$

then, since $T_2 > 0$, we get
$$\frac{\lambda-\mu}{\lambda\mu+1} < 0$$
Since $T_5 > 0$, we get
$$\frac{\lambda-\nu}{\nu\lambda+1} < 0$$
Therefore,
$$D_2 D_3 = -\frac{(\lambda-\mu)(\theta-\nu)}{(\theta-\mu)(\lambda-\nu)}\frac{(\nu\lambda+1)(\theta-\mu)}{(\mu\lambda+1)(\theta-\nu)} = -\frac{(\nu\lambda+1)(\lambda-\mu)}{(\mu\lambda+1)(\lambda-\nu)} < 0$$
and D_2 and D_3 cannot be positive at the same time.

This shows that the 4-tuple (L_1, L_2, L_3, L_4) is not positive in the sense of Fock–Goncharov for each choice of lines $l_i \in L_i$, $i \in \{1,2,3,4\}$.

5. Topology of the space of maximal representations

We now use positive $\mathcal{X}$-coordinates to understand the topology of the space of (decorated) maximal representation, focussing first on the homotopy type and then on the homeomorphism type. Note that our results are for surfaces with punctures; in the case of a closed surface, topological information about the space of maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$ can be obtained using Higgs bundles [1, 5, 11, 12].

5.1. **Homotopy type of the space of maximal representations.** Let S be an oriented surface of genus g with k punctures. In this section, we prove the following theorem

Theorem 5.1. *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is homotopically equivalent to $\mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n)$, where g is the genus of S, k is the number of punctures and the quotient is taken by the action of $\mathrm{O}(n)$ on $\mathrm{O}(n)^{2g+k-1}$ by simultaneous conjugation.*

We fix some ideal triangulation $\mathcal{T}$ of S and notice that $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) = \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.

We denote
$$\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \subseteq \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$$
the subspace of all decorated maximal representations with all edge coordinates equal to $(1,\dots,1)$. Since the edge coordinates are determined by a decorated representation, this subspace is well defined. The space $\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is one of the space of degenerate representations of constant signature, which we analyze in detail in Section 6.5. There we prove that $\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is homeomorphic to $\mathrm{O}(n)^{2g+k-1}/\,\mathrm{O}(n)$, see Corollary 6.29.

Therefore, in order to prove Theorem5.1 it is enough to prove the following lemma:

Lemma 5.2. *The space $\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ is a deformation retract of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.*

Proof. First, note that the space of positive $\mathcal{X}$-coordinates $\mathcal{X}^+(S,\mathcal{T},n)$ is by definition homeomorphic to $(\mathbb{R}^n_{>0})^{\#E} \times O(n)^{2\#F}$, where $\#E$ is the number of edges in $\mathcal{T}$ and $\#F$ is the number of triangles of $\mathcal{T}$. Since $\mathbb{R}_{>0}$ is contractible, $\mathcal{X}^+(S,\mathcal{T},n)$ is homotopy equivalent to $O(n)^{2\#F}$. We identify the space $O(n)^{2\#F}$ with the subspace

$$\{x \in \mathcal{X}^+(S,\mathcal{T},n) \mid \forall e \in E(x(e) = (1,\dots,1))\} \subset \mathcal{X}^+(S,\mathcal{T},n)$$

An explicit retraction

$$\mathcal{X}^+(S,\mathcal{T},n) \to \{x \in \mathcal{X}^+(S,\mathcal{T},n) \mid \forall e \in E(x(e) = (1,\dots,1))\}$$

is given by the formula:

$$H(x(e),x(w),t) = (1 + t(x(e)-1), x(w)), \ \ t \in [0,1]$$

for all edges e and all angles w of $\mathcal{T}$.

Since

$$[\mathrm{rep}^+]^{-1}(\mathcal{D}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))) = \\ = \{x \in \mathcal{X}^+(S,\mathcal{T},n) \mid \forall e \in E(x(e) = (1,\dots,1))\}$$

, and for all $x,x' \in \mathcal{X}^+(S,\mathcal{T},n)$ such that $[\mathrm{rep}^+](x) = [\mathrm{rep}^+](x')$ it is $[\mathrm{rep}^+](H(x,t)) = [\mathrm{rep}^+](H(x',t))$ for all $t \in [0,1]$, we can use the map $[\mathrm{rep}^+] : \mathcal{X}^+(S,\mathcal{T},n) \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$ to project the retraction H to a retraction

$$\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) \to \mathcal{D}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$$

□

As a corollary we also get

Corollary 5.3. *The space of decorated maximal representations* $\mathcal{M}^d(\pi_1(S),\mathrm{PSp}(2n,\mathbb{R}))$ *is homotopically equivalent to* $\mathrm{PO}(n)^{2g+k-1}/\,\mathrm{PO}(n)$, *where g is the genus of S, k is the number of punctures and the quotient is taken by the action of* $\mathrm{PO}(n)$ *on* $\mathrm{PO}(n)^{2g+k-1}$ *by simultaneous conjugation.*

Proof. For representations in $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{PSp}(2n,\mathbb{R}))$ all angle coordinates are in the group $\mathrm{PO}(n)$. So repeating the argument in the proof of Theorem 5.1 gives the result. □

As a corollary we obtain the following statement on the number of connected components that had been proven in [21].

Corollary 5.4. *[21, Theorem 7.2.7]*

- *The space of decorated maximal representations* $\mathcal{M}^d(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$ *has* 2^{2g+k-1} *connected components.*
- *The space of decorated maximal representations* $\mathcal{M}^d(\pi_1(S),\mathrm{PSp}(2n,\mathbb{R}))$ *has* 2^{2g+k-1} *connected components if n is even. If n is odd, it is connected.*

We now turn to determine the number of connected components of the space of maximal representation $\mathcal{M}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$ without any additional decoration. We prove the following theorem:

Theorem 5.5. *The number of connected components of* $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ *agree with the number of connected components of* $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. *In particular the space of maximal representations has* 2^{2g+k-1} *connected components.*

First, we need the following lemma:

Lemma 5.6. *Let* $M \subset \mathrm{Sp}(2n, \mathbb{R})$ *be the set of all diagonalizable symplectic matrices with pairwise different eigenvalues. Set* $M^d := \{(A, L) \in M \times \mathrm{Lag}(2n, \mathbb{R}) \mid A.L = L\}$. *Then the projection map* $p \colon M^d \to M$ *is a* $2^n : 1$-*covering map.*

Proof. Observe that since $A \in M$ has pairwise distinct real eigenvalues, it has exactly 2^n invariant Lagrangians, so the map p is a $2^n : 1$-map.

Without lost of generality, consider $A \in M$ a diagonal matrix and L some fixed Lagrangian of A. Since any small variation of A can be written as $B := T(A + \Delta)T^{-1}$ where $T \in \mathrm{Sp}(2n, \mathbb{R})$ close enough to Id and Δ is a small diagonal matrix so that $A + \Delta \in M$, we can take a small neighborhood U of A in M parameterized in this way. Since, $A + \Delta$ has distinct eigenvalues, T is well defined up to right multiplication with a matrix of the following form $\mathrm{diag}(\pm 1, \dots, \pm 1)$. These matrices act trivially on $\mathrm{Lag}(2n, \mathbb{R})$, therefore the invariant Lagrangian for B given by $T.L$ is well defined. For T small enough the rule $B \mapsto T.L$ is a continuous inverse map for $p|_U$. So p is a local homeomorphism.

The map p is a proper local homeomorphism, so it is a covering. $\square$

Remark 5.7. Let us make the following observations

- For every $A \in M$ all eigenvalues of A are different from 1. Such elements are *Shilov hyperbolic*, they have a unique attracting Lagrangian and a unique repelling Lagrangian fix point.
- The set M is an open subset of $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Sp}(2n, \mathbb{R}) \smallsetminus M$ is closed of codimension 2.

For the following discussion, we denote by $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \subset \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the space of maximal homomorphism and by $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \subset \mathrm{Hom}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ the space of decorated maximal homomorphisms, without taking conjugacy classes. Note, that the number of connected components of $\mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ is equal to the number of connected components of $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. This follows from the fact that the group $\mathrm{Sp}(2n, \mathbb{R})$ is connected. The same holds for $\mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ and $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. We denote the natural projections:

$$\Psi \colon \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \to \mathcal{M}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})),$$
$$\Psi^d \colon \mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})) \to \mathcal{M}^d(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R})).$$

Corollary 5.8. *Let* $X \subset \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ *be the subset containing all maximal representation such that for every* $\rho \in X$ *all peripheral elements* *of ρ are Shilov hyperbolic. Let X^d be the preimage of X under the projection $p\colon \mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \to \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. Then the restriction $p|_{X^d}\colon X^d \to X$ is a finite-to-one covering.*

Note that by Remark 5.7 X resp. X^d are open subsets in $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ resp. $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})))$, and the complements $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \smallsetminus X$ and $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \smallsetminus X^d$ are closed of codimension at least 2. In particular, X and $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ have the same number of connected components, and in every connected component of $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ there is a representation that is contained in X.

Proposition 5.9. *The space of maximal homomorphisms $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ has the same number of connected components as the space of decorated maximal homomorphisms.* $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.

Proof. Let N be the number of connected components of $\mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ and N_d the number of connected components of $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$.

It is immediate that $N_d \geq N$, thus we have to show that $N \geq N_d$. For this we assume that there are two decorated representations (ρ, D_1) and (ρ, D_2), which project to the same (undecorated) representations $\rho \in \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. We show that then (ρ, D_1) and (ρ, D_2) are in the same connected component of $\mathrm{Hom}^d_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. Without loss of generality we can assume that $\rho \in X$.

We consider the set of degenerate representations $\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. Note that all homomorphisms in

$$D(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) := \Psi^{-1}(\mathcal{D}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})))$$

admit only one decoration. So we can take some representation $\rho \in X$ and connect it by a path $\gamma\colon [0,1] \to \mathrm{Hom}_{\max}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ to a representation $\rho_0 \in D(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ so that $\gamma([0,1)) \subset X$.

Let (ρ, D_1), (ρ, D_2) be two lifts of ρ in $M^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. We also lift a path γ twice starting from (ρ, D_1) and from (ρ, D_2). Because of compactness of $\mathrm{Lag}(2n,\mathbb{R})$, both of these lifts finish at the same point namely at the unique lift of ρ_0 in $D^d(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$. The concatenation of these two lifted paths gives a path between (ρ, D_1) and (ρ, D_2). This proves that $N_d \leq N$. $\square$

This finishes the proof of the Theorem 5.5.

5.2. **Homeomorphism type of the space of maximal representations.** In this section we go further to determine not only the homotopy type, but actually the homeomorphism type of the space of decorated maximal representations.

We recall from the description of positive $\mathcal{X}$-coordinates, that if $\mathcal{T}$ is an idea triangulation of the oriented surface S of genus g with k punctures, the three angle coordinates associated to the three corners of one triangle satisfy the relation that their product is equal to the identity. We therefore choose in every triangle two independent angles, the third one is then uniquely defined. We denote the set of chosen independent angles by W'.

The space of positive $\mathcal{X}$-coordinates

$$\mathcal{X}^+(S,\mathcal{T},n) \cong (\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}^{n-1})^E \times \mathrm{O}(n)^{W'}$$

can be seen as a trivial bundle

$$\theta\colon \mathcal{X}^+(S,\mathcal{T},n) \to (\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}^{n-1})^E =: B$$

with compact fiber $\mathrm{O}(n)^{W'}$.

Let $y \in B$, then $y(e) = (y_1(e),\dots,y_n(e))$. Consider the set $\{y_1(e),\dots,y_n(e)\}$, let k be the cardinality of this set, so $\{y_1(e),\dots,y_n(e)\} = \{\lambda_1,\dots,\lambda_k\}$ for $\lambda_i > \lambda_{i+1}$ for all $1 \leq i \leq k-1$. We denote by n_i^e the multiplicity of λ_i in the tuple $(y_1(e),\dots,y_n(e))$.

We define the stabilizer of y to be

$$\mathrm{Stab}(y) := \prod_{e\in E} \mathrm{O}(n_1^e) \times \cdots \times \mathrm{O}(n_r^e).$$

By Proposition 4.8 the stabilizer of y acts on the fiber $\theta^{-1}(y) \subseteq \mathcal{X}^+(S,\mathcal{T},n)$ over $y \in B$. So we can consider the following singular fibration:

$$\begin{array}{ccc} \theta^{-1}(y)/\,\mathrm{Stab}(y) & \hookrightarrow & \mathcal{X}^+(S,\mathcal{T},n)/\sim \\ & & \downarrow \\ & & y \in B \end{array}$$

where the equivalence relation $\sim$ is defined fiberwise by action of $\mathrm{Stab}(y)$ on $\theta^{-1}(y) \cong \mathrm{O}(n)^{W'}$.

By proposition 4.8, the map

$$[\mathrm{rep}^+]\colon \mathcal{X}^+(S,\mathcal{T},n) \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$$

is constant on each orbit of $\mathrm{Stab}(y)$ on $\theta^{-1}(y)$. Therefore, the map

$$[\mathrm{rep}^+]' := [\mathrm{rep}^+] \circ q^{-1}\colon \mathcal{X}^+(S,\mathcal{T},n)/\sim \to \mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$$

well-defined and is a homeomorphism, where

$$q\colon \mathcal{X}^+(S,\mathcal{T},n) \to \mathcal{X}^+(S,\mathcal{T},n)/\sim$$

is the quotient map.

Since $\theta^{-1}(y) \cong \mathrm{O}(n)^{W'}$, we have the following description of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R}))$:

$$\begin{array}{ccc} \mathrm{O}(n)^{W'}/\,\mathrm{Stab}(y) & \hookrightarrow & \mathcal{M}^d_{\mathcal{T}}(\pi_1(S),\mathrm{Sp}(2n,\mathbb{R})) \\ & & \downarrow \\ & & y \in (\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}^{n-1})^E \end{array}$$

Proposition 5.10. *The space $\mathcal{X}^+(S,\mathcal{T},n)$ has $\#W' = 2\#T$ connected components that are all diffeomorphic to each other.*

The connected components of $\mathcal{X}^+(S,\mathcal{T},n)$ can be labeled by elements of the set $\{0,1\}^{W'}$.

Moreover, for each $y \in B$

$$\theta^{-1}(y) = \bigsqcup_{p \in \{0,1\}^{W'}} F_p(y)$$

where $F_p(y)$ is the fiber in the connected component C_p over $y \in B$, $p \in \{0,1\}^{W'}$. For all $y \in B$ and for all $p, q \in \{0,1\}^{W'}$ fibers $F_p(y)$ and $F_q(y)$ are diffeomorphic.

Proof. The set of connected components of $\mathcal{X}^+(S, \mathcal{T}, n)$ can be identified with the set $\{0,1\}^{W'}$, where to each independent angle w we associate 0 if it is $x(w) \in \mathrm{SO}(n)$ and 1 otherwise ($x \in \mathcal{X}^+(S, \mathcal{T}, n)$).

The diffeomorphism between connected components C_p and C_q for $p, q \in \{0,1\}^{W'}$ is given by multiplication of angle coordinates $x(w)$ with a matrix $U^{p(w)q(w)}$ for all $w \in W'$ where $U \in \mathrm{O}(n) \smallsetminus \mathrm{SO}(n)$. This diffeomorphism is given fiberwise, therefore, $F_p(y)$ and $F_q(y)$ are diffeomorphic for all $y \in B$ and for all $p, q \in \{0,1\}^{W'}$ $\square$

Proposition 5.11. *Each connected component C_p is mapped by $[\mathrm{rep}^+]$ surjectively onto some connected component of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$.*

Proof. First of all, we fix some connected component C_p and consider the restriction of $[\mathrm{rep}^+]$ to this component. $[\mathrm{rep}^+](C_p)$ is path connected and, therefore, is contained in some connected component of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ which we denote by $\mathcal{C}_p$.

Since $\theta|_{C_p} \colon C_p \to B$ is surjective, it is enough to show that $[\mathrm{rep}^+]$ maps each fiber of C_p surjectively to each fiber of $\mathcal{C}_p$ over B. We take some $y \in B$ and consider the fiber $F(y) \subseteq \mathcal{C}_p$, the fiber $F_p(y) \subseteq C_p$ and $F(y)' := \theta^{-1}(y) \cap [\mathrm{rep}^+]^{-1}(\mathcal{C}_p)$.

Since $F(y) = F(y)'/\operatorname{Stab}(y)$ is a quotient be an action of a group, the map $[\mathrm{rep}^+]|_{F(y)'} \colon F(y)' \to F(y) = F(y)'/\operatorname{Stab}(y)$ is open.

$F(y)' = \sqcup_{q \in Q} F_q(y)$ where Q is some subset in $\{0,1\}^{\#W'}$ and $p \in Q$. So $F(y)'$ is a union of finitely many diffeomorphic connected components, and $F_p(y)$ is one of them. Therefore $F_p(y)$ is open in $F(y)'$.

Moreover, since $F_p(y)$ is compact, $[\mathrm{rep}^+]F_p(y)$ is open and compact in $F(y)$, so it is closed and, therefore, $[\mathrm{rep}^+]F_p(y) = F(y)$. $\square$

Before we state the next theorem, we fix the following notation: $\mathrm{Sym}^+(n, \mathbb{R})$ is the space of all symmetric positive definite matrices,

$$\Delta^n := \{\mathrm{diag}(d_1, \dots, d_n) \mid d_1 \geq \dots \geq d_n > 0\} \subset \mathrm{Sym}^+(n, \mathbb{R}).$$

$$\operatorname{Stab}(D) := \mathrm{O}(n) \cap \mathrm{O}(D)$$

for $D \in \Delta^n$. Note that Δ^n is diffeomorphic to $\mathbb{R}_{>0} \times \mathbb{R}^{n-1}_{\geq 0}$. We freely identify edge coordinates with elements of Δ^n.

Theorem 5.12. *The space of decorated maximal representation* $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ *is homeomorphic to the singular fibration*

$$\begin{array}{ccc} F_D & \hookrightarrow & E \\ & & \downarrow \\ & & D \in \Delta^n \end{array}$$

which is obtained from the trivial bundle

$$\begin{array}{ccc} \mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1} & \hookrightarrow & \mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1} \times \Delta^n \\ & & \downarrow \\ & & D \in \Delta^n \end{array}$$

by dividing fiberwise by the action by common conjugation of $\mathrm{Stab}(D)$ *on the fiber over* $D \in \Delta^n$*, i.e.*

$$F_D = \left(\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}\right) / \, \mathrm{Stab}(D)$$

where $\mathrm{Stab}(D)$ *acts on* $\mathrm{Sym}^+(n,\mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}$ *by common conjugation.*

Proof. We consider a special ideal triangulation of S, see Figure 5.1.

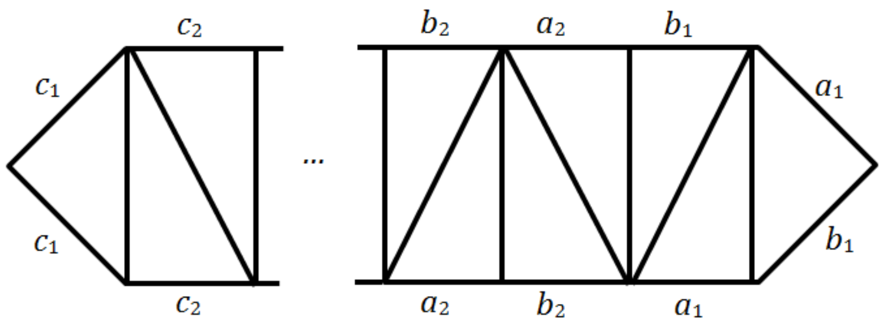

FIGURE 5.1. Triangulation of S. Sides with the same labels are identified

This triangulation divides the surface in blocks of four different types. The blocks of type 1, see Figure 5.2, the clock of type 2, see Figure 5.4, and the blocks of type 3 and 4, see Figure 5.6.

We parametrize each block and then describe, how to glue the different blocks together. Recall that we chose two independent angles in each triangle, the third angles coordinate is then uniquely determined.

Block of type 1: We choose independent angles as indicated in Figure 5.2, with coordinates $U_1, ..., U_6$ and denote by D_0, D_1, D_2, D_3 the edge coordinates (considered as diagonal $n \times n$-matrices, where the entries are ordered by size).

We define three maps:

$$f_1(U_1, D_1, U_2) = (U_1 D_1 U_1^{-1}, U_1 U_2^{-1}) =: (S_1, V_1),$$

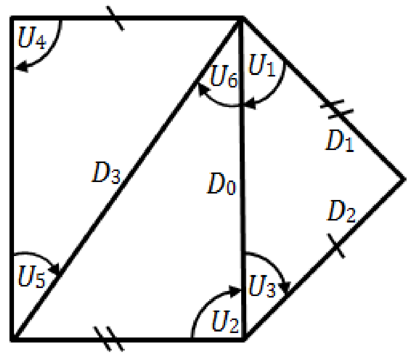

FIGURE 5.2. Block of type 1

$$f_2(U_3, D_2, U_4) = (U_3^{-1} D_2 U_3, U_3^{-1} U_4^{-1}) =: (S_2, V_2),$$
$$f_3(U_5, D_3, U_6) = (U_5^{-1} D_3 U_5, U_5^{-1} U_6) =: (S_3, V_3),$$

where S_i are symmetric matrices, and V_i are orthogonal matrices. By definition, these maps are invariant under changing of angles along edges with coordinates D_1, D_2, D_3. We consider (S_i, V_i) as new coordinates on the block of type 1 (see Figure 5.3 left).

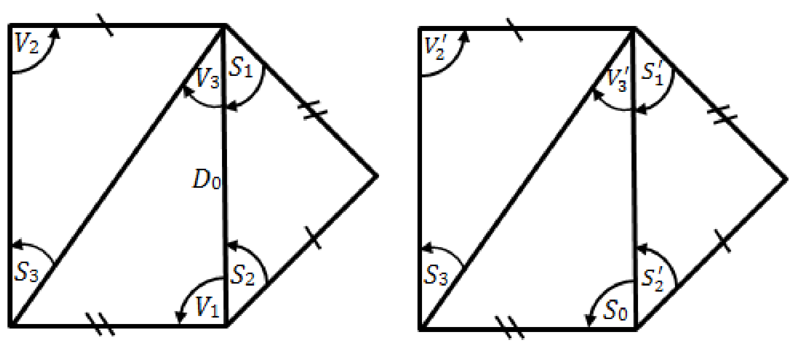

FIGURE 5.3. New coordinates on the block of type 1

From the remaining "unused" edge coordinate D_0 we get an additional equivalence relation for the new coordinates $\{(S_i, V_i)\}$ (see Proposition 4.8). We could multiply the angle coordinates U_1, U_6, U_2, U_3 by elements of $\mathrm{Stab}(D_0)$. This induces the following equivalence relation:

$$S_1 \sim W S_1 W^{-1}$$
$$S_2 \sim W S_2 W^{-1}$$
$$V_1 \sim V_1 W^{-1}$$
$$V_2 \sim W V_2$$

$$V_3 \sim V_3 W^{-1}$$

for $W \in \mathrm{Stab}(D_0)$. We therefore define the map:

$$f_4(S_1, S_2, V_1, V_2, V_3, D_0) :=$$
$$= (V_1 S_1 V_1^{-1}, V_1 S_2 V_1^{-1}, V_1 D_0 V_1^{-1}, V_1 V_2, V_3 V_1^{-1}) =$$
$$=: (S_1', S_2', S_0, V_2', V_3').$$

By definition, these maps are invariant under changing of angles along the edges with coordinates D_0. We consider $(S_0, S_1', S_2', S_3, V_2', V_3')$ as new coordinates on the block of type 1 (see Figure 5.3 right). They define the old edge and angle coordinates exactly up to equivalence relation given by Proposition 4.8.

Note, that we have not yet used the left edge. this edge will play a role when gluing the different blocks . Changing of angle coordinates along this edge induces a global conjugation on all new coordinates of the block of type 1.

Block of type 2: We now proceed in a similar way, we choose independent angles as indicated in Figure 5.4 with coordinates $U_1, ..., U_8$ and denote by D_0, D_1, D_2, D_3, D_4 the edge coordinates.

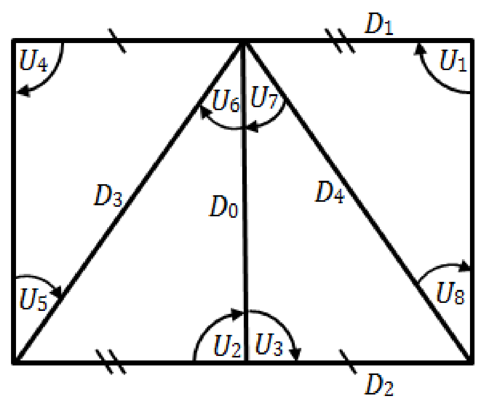

FIGURE 5.4. Block of type 2

We introduce new coordinates (S_i, V_i) on the block of type 2 (see Figure 5.5 left) by defining

$$f_1(U_1, D_1, U_2) = (U_2 D_1 U_2^{-1}, U_1^{-1} U_2^{-1}) =: (S_1, V_1),$$
$$f_2(U_3, D_2, U_4) = (U_3^{-1} D_2 U_3, U_3^{-1} U_4^{-1}) =: (S_2, V_2),$$
$$f_3(U_5, D_3, U_6) = (U_5^{-1} D_3 U_5, U_5^{-1} U_6) =: (S_3, V_3),$$
$$f_4(U_7, D_3, U_8) = (U_7 D_4 U_7^{-1}, U_7 U_8^{-1}) =: (S_4, V_4).$$

By definition, these maps are invariant under changing of angles along edges with coordinates D_1, D_2, D_3, D_4.

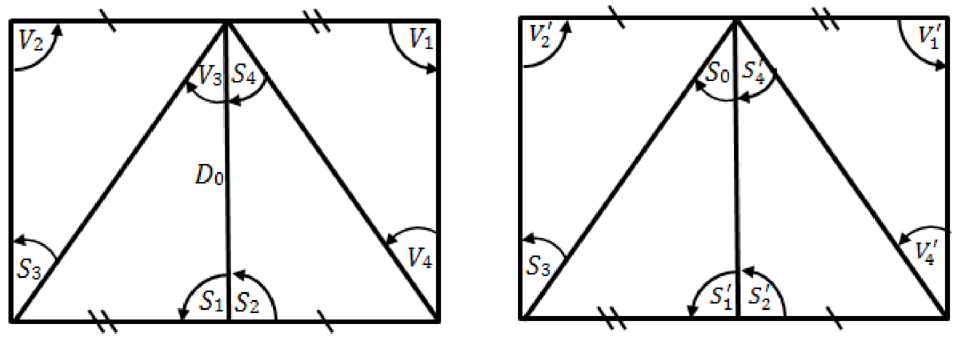

Figure 5.5. New coordinates on the block of type 2

The "unused" edge with coordinate D_0 gives us an additional equivalence relation We could multiply U_7, U_6, U_2, U_3 by elements of $\mathrm{Stab}(D_0)$. This induces the following equivalence relation:

$$S_1 \sim W S_1 W^{-1}$$
$$S_2 \sim W S_2 W^{-1}$$
$$S_4 \sim W S_4 W^{-1}$$
$$V_1 \sim V_1 W^{-1}$$
$$V_2 \sim W V_2$$
$$V_3 \sim V_3 W^{-1}$$
$$V_4 \sim W V_4$$

for $W \in \mathrm{Stab}(D_0)$. Therefore we set :

$$f_4(S_1, S_2, S_4, V_1, V_2, V_3, V_4, D_0) :=$$
$$= (V_3 S_1 V_3^{-1}, V_3 S_2 V_3^{-1}, V_3 S_4 V_3^{-1}, V_3 D_0 V_3^{-1}, V_1 V_3^{-1}, V_3 V_2, V_3 V_4) =$$
$$=: (S_1', S_2', S_4', S_0, V_1', V_2', V_4').$$

and consider $(S_0, S_1', S_2', S_3, S_4', V_1', V_2', V_4')$ as a new coordinates on the block of type 2 (see Figure 5.5 right). They define the old edge and angle coordinates exactly up to equivalence relation given by Proposition 4.8.

Block of type 3: We choose independent angles as indicated in Figure 5.6 left, with coordinates $U_1, ..., U_4$ and denote by D_1, D_2 the edge coordinates. Consider

$$f_1(U_1, D_1, U_2) = (U_2^{-1} D_1 U_2, U_1^{-1} U_2) =: (S_1, V_1),$$
$$f_2(U_3, D_2, U_4) = (U_4 D_2 U_4^{-1}, U_3 U_4^{-1}) =: (S_2, V_2).$$

By definition, these maps are invariant under changing of angles along edges with coordinates D_1, D_2, and we consider (S_i, V_i) as a new coordinates on the block of type 3 (see Figure 5.7 left).

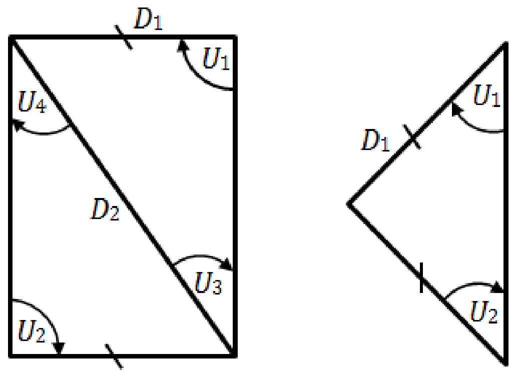

FIGURE 5.6. Block of type 3 (left), block of type 4 (right)

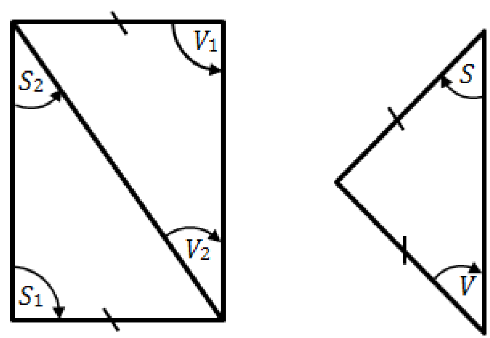

FIGURE 5.7. New coordinates on the block of type 3 (left) and on the block of type 4 (right)

Block of type 4: We choose independent angles as indicated in Figure 5.6 right, with coordinates U_1, U_2 and denote by D_1 the edge coordinate. We define

$$f(U_1, D_1, U_2) = (U_1^{-1} D_1 U_1, U_2 U_1) =: (S, V),$$

and consider (S, V) as a new coordinates on the block of type 4 (see Figure 5.7 right). Note, the right edge which we have not used yet will play a role in the gluing of blocks. Changing of angle coordinates along this edge induces a global conjugation of the new coordinate.

With this, for every block we have now a parametrization given by several copies of $\mathrm{Sym}^+(n, \mathbb{R})$ and of orthogonal groups $\mathrm{O}(N)$. We now explain how to glue the different blocks.

We will glue blocks from the right to the left as on the Figure 5.1 by induction. Assume that the part of the surface laying to left has the parametrization $\mathcal{P}_l = \mathrm{O}(n) \times \mathrm{O}(n) \times \mathcal{P}'_r$, the block lying to the right has parametrization $\mathcal{P}_r = \mathrm{Sym}^+(n, \mathbb{R})^{N_1} \times \mathrm{O}(n)^{N_2}$ for some $N_1, N_2 > 0$ and this is not the last step of gluing so it is not the block of the type 4. We assume as well that changing of angles around the gluing edge by an angle $W \in \mathrm{Stab}(D)$ induces a conjugation of all coordinates in $\mathcal{P}_l$ by W. We can assume that this holds by induction, since in the first step, when gluing a block of type 1 with some other block it holds.

We describe the gluing of two blocks along an edge with coordinate D and coordinates around this edge as in Figure 5.8. We denote by K_i the coordinates in $\mathcal{P}_l$.

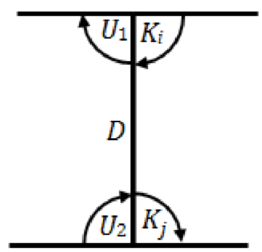

FIGURE 5.8. Gluing, intermediate step

The edge with coordinate D gives us an additional equivalence relation for coordinates $\mathcal{P}_l$ and (U_1, U_2):

$$K_i \sim W K_i W^{-1}$$
$$U_1 \sim U_1 W^{-1}$$
$$U_2 \sim W U_2$$

for $W \in \mathrm{Stab}(D)$ and for all K_i coordinates of $\mathcal{P}_r$. So we can define the map:

$$f_{gl}(U_1, U_2, (K_i), D) := (U_1 U_2, U_1 D U_1^{-1}, (U_1 K_i U_1^{-1})).$$

By definition, this map is invariant under changing of angles along edges with coordinates D. We consider these as the new coordinates on the glued block. They define the old coordinates exactly up to equivalence relation given by Proposition 4.8. Note, that there is a right edge which we have not used yet. Changing of angle coordinates along this edge induces conjugation on the new coordinate of glued block.

Now we describe the last step of gluing with a block of type 4. We can write again $\mathcal{P}_r = \mathrm{Sym}^+(n, \mathbb{R})^{N_1} \times \mathrm{O}(n)^{N_2}$ for some $N_1, N_2 > 0$ and $\mathcal{P}_l = \mathrm{Sym}^+(n, \mathbb{R}) \times \mathrm{O}(n)$. Coordinates on the glued edge is D (see Figure 5.9).

As we have seen, the changing of angles around this edge by some $W \in \mathrm{Stab}(D)$ induces the common conjugation by W of all coordinates in $\mathcal{P}_r$ and $\mathcal{P}_l$. To define the space which is in 1-1 correspondence with the $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$ we have to take a quotient by conjugation depending on D. It can be seen as a singular fibration coming from the projection map:

$$p \colon \mathcal{P} \to \Delta^n$$

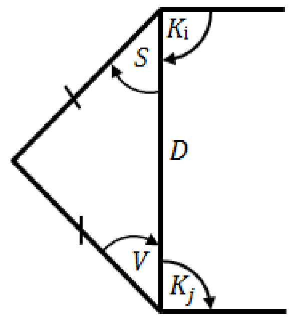

FIGURE 5.9. Gluing, last step

of $\mathcal{P} := \mathcal{P}_r \times \mathcal{P}_l \times \Delta^n$ to Δ^n by dividing of the equivalence relation $\sim$ such that for each $K, K' \in \mathcal{P}$ with $p(K) = p(K')$ it is $K \sim K'$ if and only if $(K'_i) = (WK_iW^{-1})$ for some $W \in \mathrm{Stab}(p(K))$, where $K = (K_i), K' = (K'_i)$. □

Remark 5.13. From the last theorem we get:

$$\mathcal{P} = \mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-7} \times \mathrm{O}(n)^{2g+k-1} \times \Delta^n.$$

The dimension of $\mathcal{P}$ agrees with the dimension of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, R))$, so we get:

$$\dim(\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))) = (2g+k-2)n(2n+1) = |\chi(S)| \dim(\mathrm{Sp}(2n, \mathbb{R})).$$

Remark 5.14. Consider the subset

$$\Delta^n_{gen} = \{\mathrm{diag}(d_1, \dots, d_n) \mid \forall i \in \{1, \dots, n-1\}(d_i \neq d_{i+1})\} \subset \Delta^n$$

then for all $D \in \Delta^n_{gen}$ it is $\mathrm{Stab}(D) = \mathrm{O}(1)^n$. We can consider the subfibration $E_0 := E|_{\Delta^n_{gen}} \to \Delta^n_{gen}$. Since $\mathrm{Stab}(D) = \mathrm{O}(1)^n$ for all $D \in \Delta^n_{gen}$, we have

$$E_0 = \left(\left(\mathrm{Sym}^+(n, \mathbb{R})^{6g+3k-6} \times \mathrm{O}(n)^{2g+k-1}\right) / \mathrm{O}(1)^n\right) \times \Delta^n_{gen}$$

where $\mathrm{O}(1)^n \leq \mathrm{O}(n)$ acts by simultaneous conjugation. This is an orbifold and it is an open dense subset of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$.

Remark 5.15. The definition of $E_0 =: E(e_0)$ depends on the edge e_0 along which we were gluing in the last step in the proof of the Theorem 5.12. Actually, we can choose any edge to do this last gluing. So for each edge e the constructed as above subspace $E(e)$ is homeomorphic to E_0. Because the property to be an orbifold is a local property, the finite union of all $E(e)$ for all edges e is an orbifold. We denote this subspace by E' and call it *generic part* of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$. It contains all representation with at least one edge coordinate in Δ^n_{gen}. This is an open dense subset of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n, \mathbb{R}))$.

Corollary 5.16. *The space E' for $n = 2$ contains all Zariski dense representations.*

Proof. Let $[\rho, D] \in \mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \smallsetminus E'$ and $x \in \mathcal{X}^+(S,\mathcal{T},n)$ such that $[\mathrm{rep}^+](x) = [\rho, D]$. Then for every edge e, $x(e) = (\lambda,\lambda)$ for some $\lambda > 0$. By Proposition 6.23, $[\rho, D]$ is a representation into some copy by conjugation of $\mathrm{SL}(2,\mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(2) \leq \mathrm{Sp}(4,\mathbb{R})$, therefore, it is not Zariski dense. □

5.2.1. *Topology of* $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$. Theorem 5.12 give a description of the homeomorphism type of the space of decorated maximal representations $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ as a singular fibration. When $n = 2$ we can go further to explicitly determine the homeomorphism type of all connected components of $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(4,\mathbb{R}))$. In this case the singular fibration is

$$(\mathrm{Sym}^+(2,\mathbb{R})^N \times \mathrm{O}(2)^M \times \Delta^2)/\sim \to \Delta^2$$

where $N = 6g+3k-7$, $M = 2g+k-1$ and the equivalence relation $\sim$ is given by fiberwise action of the group $\mathrm{Stab}(y)$ for $y \in \Delta^2$. Since $n = 2$, there are two possibilities for the stabilizer , we can have $\mathrm{Stab}(y) = \mathrm{O}(1)\times\mathrm{O}(1) < \mathrm{O}(2)$ when $y = (d_1, d_2)$ with $d_1 \neq d_2$, and $\mathrm{Stab}(y) = \mathrm{O}(2)$ for $y = (d, d)$. Since $\mathrm{Stab}(y)$ acts by simultaneous conjugation on all factors, there is a kernel $\{\pm \mathrm{Id}\} \in \mathrm{O}(1) \times \mathrm{O}(1)$ of this action.

We identify $\mathrm{Sym}^+(2,\mathbb{R})$ using the polar coordinates with $\mathbb{R}_{>0} \times \mathbb{C}$. Then the action of $g \in \mathrm{SO}(2)$ is a rotation in $\mathbb{C}$-factor, for $g = \mathrm{diag}\{1,-1\}$ it is the reflection around the x-axis. Since $\mathrm{O}(2) = \mathbb{Z}_2 \ltimes \mathrm{SO}(2)$ where $\mathbb{Z}_2 = \{\mathrm{Id}, \mathrm{diag}(1,-1)\}$, first we can quotient out the fiberwise action of $\mathrm{Stab}(y) \cap \mathrm{SO}(2)$ and then the global action of $\mathbb{Z}_2$.

We now focus first on analyzing the connected component $\mathcal{C}_0 := (\mathrm{Sym}(2,\mathbb{R})^N \times \mathrm{SO}(2)^M \times \Delta^2)/\sim$.

Theorem 5.17. *The connected component* C_0 *is homeomorphic to the product* $\mathbb{R}^{N+1}_{>0} \times Q$*, where* $(Q = (S^1)^M \times Q_1)/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *acts by the diagonal complex conjugation on each factor.* $Q_1 = (\mathbb{C}^N \times \mathbb{R}_{\geq 0})/\sim_1 \to \mathbb{R}_{\geq 0}$ *is a singular fibrations, whose total space is equal to* $\mathbb{C}^N \times \mathbb{R}_{>0} \sqcup \mathbb{C}^N/\mathrm{SO}(2) \times \{0\}$. *In particular* Q_1 *is a manifold away from* $(0,\ldots,0) \in \mathbb{C}^N \times \mathbb{R}_{\geq 0}$*, and* $(0,\ldots,0)$ *is not an orbifold point.*

We subdivide the proof of Theorem 5.17 into several Lemmata.

First note that we can write

$$\mathcal{C}_0 = Q_0/\mathbb{Z}_2$$
$$Q_0 := (\mathrm{Sym}^+(2,\mathbb{R})^N \times \mathrm{SO}(2)^M \times \Delta^2)/\sim'$$

where $\sim'$ is the equivalence relation defined by the fiberwise action of $\mathrm{Stab}(y) \cap \mathrm{SO}(2)$ by simultaneous conjugation, and then $\mathbb{Z}_2$ acts by simultaneous conjugation by $\mathrm{diag}(1,-1)$.

Lemma 5.18. Q_0 *is homeomorphic to the product of* $\mathbb{R}^{N+1}_{>0} \times (S^1)^M$ *and the singular fibration* $Q_1 := (\mathbb{C}^N \times \mathbb{R}_{\geq 0})/\sim_1 \to \mathbb{R}_{\geq 0}$ *where* $\sim_1$ *is defined fiberwise: if* $0 \neq x \in \mathbb{R}_{\geq 0}$ *then* $\sim_1$ *is trivial; if* $x = 0$ *then* $\sim_1$ *is given by the diagonal* $\mathrm{SO}(2)$*-action by rotations on* $\mathbb{C}^N$ *around the origin. We can write:*

$$(\mathbb{C}^N \times \mathbb{R}_{\geq 0})/\sim_1 = \mathbb{C}^N \times \mathbb{R}_{>0} \sqcup \mathbb{C}^N/\mathrm{SO}(2) \times \{0\}.$$

This fibration is a manifold everywhere except for the point $(0,\dots,0,0) \in \mathbb{C}^N \times \mathbb{R}_{\geq 0}$. *We denote* $Q_1' := Q_1 \setminus \{(0,\dots,0,0)\}$.

Proof. The homeomorphism of Q_0 with the product of $\mathbb{R}_{>0}^{N+1} \times (S^1)^M$ and the singular fibration $Q_1 = (\mathbb{C}^N \times \mathbb{R}_{\geq 0})/\sim_1 \to \mathbb{R}_{\geq 0}$ is given just by the identification above of $\mathrm{Sym}^+(2,\mathbb{R})$ with $\mathbb{R}_{>0} \times \mathbb{C}$, $\mathrm{SO}(2)$ with $U(1) = S^1 \subset \mathbb{C}$ and Δ^2 with $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$.

The generic part of Q_0 projects to points $x \in Q_1$ such that $x \neq (0,\dots,0,0)$. Since the generic part of Q_0 is a manifold, $Q_1' = Q_1 \setminus \{(0,\dots,0,0)\}$ is a manifold. □

Lemma 5.19. *The connected component* $\mathcal{C}_0$ *is homeomorphic to the product of* $\mathbb{R}_{>0}^{N+1}$ *and the quotient* $Q_2 := ((S^1)^M \times Q_1)/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *acts by the diagonal complex conjugation on each factor.*

Let $Q_2' := ((S^1)^M \times Q_1')/\mathbb{Z}_2 \subset Q_2$. Q_2' *is a manifold everywhere except for points of the following form:* $(s_1,\dots,s_M,[z_1,\dots,z_N,r])$, *where all* $s_i \in \{\pm 1\}$, $z_i \in \mathbb{R}$, $r \geq 0$.

Proof. Since for $r = 0$ we have $(z_1,\dots,z_2) \sim_1 (z_1 e^{i\phi},\dots,z_n e^{i\phi})$ for every $\phi \in \mathbb{R}$, it is

$$(\bar{z}_1,\dots,\bar{z}_2) \sim_1 (\bar{z}_1 e^{-i\phi},\dots,\bar{z}_n e^{-i\phi}) = (\overline{z_1 e^{i\phi}},\dots,\overline{z_n e^{i\phi}}),$$

the complex conjugation on Q_1 is well-defined. This gives the homeomorphism given in the statement of the lemma.

Since the action by diagonal conjugation is free and discrete everywhere on $(S^1)^M \times Q_1'$ except for real points, the corresponding quotient is a manifold. □

Lemma 5.20. Q_2' *is an orbifold but not a manifold. The real points of* Q_2' *are orbifold points. Small neighborhoods of these points are homeomorphic to products of Euclidian balls of dimension* $N+1$ *and Euclidian balls of dimension* $M+N$ *modulo the antipodal map.*

Proof. Let $p := (s_1,\dots,s_M,x_1,\dots,x_N,r) \in (S^1)^M \times Q_1'$ be some real point. Without loss of generality, we can assume $r > 0$. Then

$$p \in (S^1)^M \times (\mathbb{C}^N \times \mathbb{R}_{>0})/\sim_1 = (S^1)^M \times \mathbb{C}^N \times \mathbb{R}_{>0}.$$

Note that $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ and since $\mathbb{Z}_2$ acts by complex conjugation on $(S^1)^M \times \mathbb{C}^N \times \mathbb{R}_{>0}$, we can write:

$$(S^1)^M \times \mathbb{C}^N \times \mathbb{R}_{>0}/\mathbb{Z}_2 \cong ((S^1)^M \times \mathbb{R}^N)/\mathbb{Z}_2 \times \mathbb{R}^N \times \mathbb{R}_{>0}$$

where $\mathbb{Z}_2$ acts on $\mathbb{R}$-factors by antipodal map. In a small neighborhood $U_\pm$ of $\pm 1 \in S^1$ the map $U_\pm \ni \pm e^{it} \mapsto t \in V = (-\varepsilon,\varepsilon)$ is a homeomorphism. $\mathbb{Z}_2$-action by conjugation on $U_\pm$ induces the action by antipodal map on V. The fixed points by this $\mathbb{Z}_2$-action are exactly the real points

Note that $N+M > 3$. The fact that Q_2' is not a manifold follows from

Proposition 5.21. *Let X be a smooth manifold, G be a finite group acting on X by diffeomorphisms. Let X' be the subset of X consisting of points with non-trivial stabilizer in G. Assume, X' is discrete in X.*

If $\dim X \geq 3$, then X/G is not a topological manifold, but $(X \setminus X')/G$ is a smooth manifold.

Proof of Proposition. We prove it by contradiction. Assume, X/G is a manifold.

Note, the quotient map $q\colon X \to X/G$ is open because G acts by diffeomorphism on X. Moreover, $q|_{X \setminus X'}$ is a covering map. Therefore, $(X \setminus X')/G$ is a manifold

Let $x \in X'$ and $y := q(x)$. Since X/G assumed to be a topological manifold, we can take an open neighborhood V of y which is homeomorphic to an Euclidian ball. Then $q^{-1}(V)$ is a union of open sets which are open neighborhoods of points of $q^{-1}(y)$. We can always assume that it is a disjoint union of neighborhoods of points in $q^{-1}(y)$ by taking V small enough.

We take a component U of $q^{-1}(V)$ which is an open connected neighborhood of x. We can take $V' \subset q(U)$ open neighborhood of $y \in X/G$ homeomorphic to an Euclidian ball because q is open. Then $U' := q|_U^{-1}(V')$ is connected, open in X and $q(U') = V' = U'/G_x$, where G_x is the stabilizer of x in G.

The group G_x acts freely and properly discontinuously on $U' \setminus \{x\}$. Therefore, $G \leq \pi_1((U' \setminus \{x\})/G_x) = \pi_1(V' \setminus \{y\}) \neq \{1\}$, but $V' \setminus \{y\}$ is a Euclidian ball without one point of dimension at least 3, so it has a trivial fundamental group. This is a contradiction to the assumption that X/G is a manifold. □

Remark 5.22. The condition $\dim X \geq 3$ is essential. To see it, take $X = S^1 \times S^1 \subseteq \mathbb{C}^2$ and $G = \mathbb{Z}_2 = \{1, a\}$ and $a(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Then X/G is homeomorphic to S^2, so it is a manifold.

□

To prove that the point $0 = (0, \ldots, 0) \in Q_1 = \mathbb{C}^N \times \mathbb{R}_{>0} \sqcup \mathbb{C}^N / \operatorname{SO}(2) \times \{0\}$ is not an orbifold singularity we use the following proposition (see [17, Exercise 3.3.33]):

Proposition 5.23. *If M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere; that is $H_i(\partial M; \mathbb{Z}) \cong H_i(S_{n-1}; \mathbb{Z})$ for all i.*

We note, that Q_1 is a manifold everywhere except for 0 because $\mathbb{C}^N \times \mathbb{R}_{>0} \subset Q_1$ is an open manifold, and as a fibration $Q_1 \to \mathbb{C}^N / \operatorname{SO}(2)$ it is a cone bundle everywhere except for $0 \in \mathbb{C}^N / \operatorname{SO}(2)$. Since $\mathbb{C}^N / \operatorname{SO}(2) \setminus \{0\} \cong \mathbb{R}_{>0} \times \mathbb{C}P^{N-1}$ is topologically a manifold, the corresponding cone bundle is a topological manifold.

First of all, we take an contractible neighborhood of 0 in Q_1 of the following form:

$$U := B \times [\varepsilon, 0) \sqcup B / \operatorname{SO}(2) \times \{0\}$$

where $B = \{z \in \mathbb{C}^N \mid \|z\| \leq \varepsilon\}$ for some $\varepsilon > 0$. Everywhere except for 0 it is a manifold with boundary

$$\partial U = B \times \{\varepsilon\} \sqcup \partial B \times (\varepsilon, 0) \sqcup \partial B / \operatorname{SO}(2) \times \{0\}$$

where $\partial B \cong S^{2N-1}$. ∂U is simply connected, so if we assume $0 \in Q_1$ to be an orbifold point, then, by Proposition ∂U, have to be a finite quotient of homology sphere, but by generalized Poincaré conjecture, ∂U have to be a sphere since it is simply connected. It contradicts to the fact that $\partial U \smallsetminus \{0\} \times \{\varepsilon\}$ contracts to the complex projective space

$$\mathbb{C}P^{N-1} \cong \partial B / \operatorname{SO}(2) \times \{0\}$$

that is not contractible.

Now we consider any of the other connected components

$$\mathcal{C}_q := (\operatorname{Sym}^+(2,\mathbb{R})^N \times \operatorname{SO}(2)^{M-q} \times (J \operatorname{SO}(2))^q \times \Delta^2) / \sim$$

where $J = \operatorname{diag}(1,-1)$, $q \neq 0$. We prove

Theorem 5.24. *The connected component $\mathcal{C}_q$ is homeomorphic to*

$$\mathbb{R}_{>0}^{N+1} \times \mathbb{R}^{N+1} \times ((S^1)^{M-1} \times \mathbb{R}^{N+1}) / \mathbb{Z}_2.$$

$((S^1)^{M-1} \times \mathbb{R}^{N+1})/\mathbb{Z}_2$ is a manifold everywhere except for the following points: $(\pm 1, \dots, \pm 1, 0, \dots, 0)$. These points are orbifold points. Small neighborhoods of them are homeomorphic to Euclidian balls modulo the antipodal map.

We can write

$$\mathcal{C}_q = Q_q / \mathbb{Z}_2$$
$$Q_q := (\operatorname{Sym}^+(2,\mathbb{R})^N \times \operatorname{SO}(2)^{M-q} \times (J \operatorname{SO}(2))^q \times \Delta^2) / \sim'$$

where $\sim'$ is the equivalence relation defined by the fiberwise action of $\operatorname{Stab}(y) \cap \operatorname{SO}(2)$ by simultaneous conjugation, and then $\mathbb{Z}_2$ acts by simultaneous conjugation by $\operatorname{diag}(1,-1)$.

Then Theorem 5.24 is a direct consequence of the following

Lemma 5.25. *Q_q is homeomorphic to*

$$\mathbb{R}_{>0}^{N+1} \times \mathbb{R}^{N+1} \times (S^1)^{M-1} \times \mathbb{R}^{N+1}.$$

Proof. As before, we identify $\operatorname{Sym}^+(2,\mathbb{R}) \cong \mathbb{R}_{>0} \times \mathbb{C}$, $\Delta^2 \cong \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$, $\operatorname{SO}(2) \cong S^1$. We also identify $J \operatorname{SO}(2)$ with $\operatorname{SO}(2) \cong S^1$ by the map $JU \mapsto U$ and write:

$$Q_q \cong \mathbb{R}_{>0}^{N+1} \times (S^1)^{M-q} \times (\mathbb{C}^N \times (S^1)^q \times \mathbb{R}_{\geq 0}) / \sim_1$$

where $\sim_1$ is trivial for $0 \neq x \in \mathbb{R}_{\geq 0}$ and is given by the diagonal action of $\operatorname{SO}(2)$ by rotations on S^1-factors and $\mathbb{C}$-factors around the origin.

Since $q \neq 0$ we can consider the following map:

$$f \colon \mathbb{C}^N \times (S^1)^q \times \mathbb{R}_{\geq 0} \to \mathbb{C}^N \times (S^1)^q \times \mathbb{R}_{\geq 0},$$

$$f(z_1, \dots, z_N, s_1, \dots, s_q, r) := (z_1 s_q^{-1}, \dots, z_N s_q^{-1}, s_1 s_q^{-1}, \dots, s_{q-1} s_q^{-1}, s_q, r).$$

This map is a homeomorphism, and the first $N+q-1$ components are invariant under $\sim_1$. So we can write:

$$Q_q \cong \mathbb{R}_{>0}^{N+1} \times (S^1)^{M-1} \times \mathbb{C}^N \times (S^1 \times \mathbb{R}_{\geq 0})/\sim_2$$

where $\sim_2$ is trivial for $0 \neq x \in \mathbb{R}_{\geq 0}$ and is given by the action of $\mathrm{SO}(2)$ by rotations on S^1. So using the polar coordinates, we identify $(S^1 \times \mathbb{R}_{\geq 0})/\sim_2 \cong \mathbb{C}$ and get

$$Q_q \cong \mathbb{R}_{>0}^{N+1} \times (S^1)^{M-1} \times \mathbb{C}^{N+1}$$

Since $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ we can write:

$$Q_q \cong \mathbb{R}_{>0}^{N+1} \times \mathbb{R}^{N+1} \times (S^1)^{M-1} \times \mathbb{R}^{N+1}$$

□

6. General $\mathcal{X}$-coordinates

In this section we introduce general, not necessarily positive $\mathcal{X}$-coordinates with respect to a chosen ideal triangulation $\mathcal{T}$ of S. General $\mathcal{X}$-coordinates consists of triangle invariants, which are signatures of certain quadratic forms, associated to every triangle of $\mathcal{T}$, edge invariants and angle invariants.

For the edge invariants we had to simultaneously diagonalize pairs of positive definite bilinear forms. Here we would have to simultaneously diagonalize pairs of non-degenerate bilinear forms of varying signature. This is in general impossible. We need to find some analog of this diagonalization process. To do this, we use the following theorem (for the proof and details, see Appendix A.3):

Theorem 6.1. *Let β_3, β_4 be two symmetric non-degenerate bilinear forms on some vector space L. We consider β_3, β_4 as maps $L \to L^*$ and define the map $\phi := \beta_3^{-1} \circ \beta_4$.*

Then there exists a basis $\mathbf{e}$ of L such that

$$[\phi]_e = X^0(\beta_3, \beta_4) := \begin{pmatrix} \mathcal{J}_1 & 0 & 0 \\ 0 & \mathcal{J}_2 & 0 \\ 0 & 0 & \mathcal{K} \end{pmatrix}$$

$$[\beta_3]_e = X^1(\beta_3, \beta_4) := \begin{pmatrix} \mathcal{I}_1^* & 0 & 0 \\ 0 & -\mathcal{I}_2^* & 0 \\ 0 & 0 & \mathcal{I}^{2*} \end{pmatrix}$$

$$[\beta_4]_e = X^2(\beta_3, \beta_4) := X^1(\beta_3, \beta_4) X^0(\beta_3, \beta_4)$$

where for $r = 1, 2$

$$\mathcal{I}_r^* = \begin{pmatrix} I_{1r}^* & 0 & \dots & 0 & 0 \\ 0 & I_{2r}^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_{k_r r}^* \end{pmatrix}, \quad \mathcal{J}_r = \begin{pmatrix} J_{1r} & 0 & \dots & 0 & 0 \\ 0 & J_{2r} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_{k_r r} \end{pmatrix}$$

$$\mathcal{I}^{2*} = \begin{pmatrix} I_1^{2*} & 0 & \dots & 0 & 0 \\ 0 & I_2^{2*} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_s^{2*} \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} K_1 & 0 & \dots & 0 & 0 \\ 0 & K_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & K_s \end{pmatrix}$$

where $n_{ir} := \dim(I_{ir}^*) = \dim(J_{ir})$, $m_j := \dim(I_j^{2*}) = \dim(K_j)$ *and*

$$I_{ir}^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_{ir} \times n_{ir}}$$

$$I_j^{2*} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ & & & & \dots & & \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{m_j \times m_j}$$

J_{ir} *are Jordan blocks with eigenvalue* $\lambda_{ir} \in \mathbb{R}$, K_j *are generalized Jordan blocks with eigenvalue* $\mu_j \in \mathbb{C} \setminus \mathbb{R}$ *such that* $\lambda_{ir} \geq \lambda_{i+1,r}$, $\mu_j \geq \mu_{j+1}$, *where for complex numbers the following linear order is used:* $x + iy > x' + iy'$ *if* $x > x'$ *or* $x = x'$ *and* $y > y'$.

Remark 6.2. The basis $\mathbf{e}$ is in general not unique but the matrices $X^0(\beta_3, \beta_4)$, $X^1(\beta_3, \beta_4)$, $X^2(\beta_3, \beta_4)$ are well defined by β_3, β_4. We denote the edge invariant by

$$X(\beta_3, \beta_4) := (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$$

The triple $X(\beta_3, \beta_4)$ defines $X^0(\beta_3, \beta_4)$, $X^1(\beta_3, \beta_4)$, $X^2(\beta_3, \beta_4)$ uniquely.

Definition 6.3. The *signature* of the triple $X(\beta_3, \beta_4) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ is the signature of the bilinear form $X^0(\beta_3, \beta_4)$. We will write $\operatorname{sgn}(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$. This is the triangle invariant.

Definition 6.4. We denote by $\mathcal{E}(n)$ the set of all triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}$ are of the form as in the Theorem 6.1 with

$$\dim \mathcal{J}_1 + \dim \mathcal{J}_2 + \dim \mathcal{K} = n$$

Definition 6.5. If the basis $\mathbf{e}$ of L is chosen so that $[\beta_3]_{\mathbf{e}} = X^1(\beta_3, \beta_4)$, $[\beta_4]_e = X^2(\beta_3, \beta_4)$, we will say that in the basis $\mathbf{e}$ the pair of forms (β_3, β_4) is in the *standard form.*

6.1. **The angle of five Lagrangians.** To define the angel invariant in general $\mathcal{X}$-coordinates we need an invariant of 5 Lagrangians. In the Section 2 we already defined this invariant only in the case when all triangles have maximal Maslov index. Now we do it in the general case.

For the 4-tuple (L_1, L_2, L_3, L_4) there exists a basis $\mathbf{e}$ of L_1 such that two bilinear forms $\beta_0 := [L_1, L_3, L_2]$ and $\beta_0' := [L_1, L_4, L_2]$ are in the standard form, i.e. $[\beta_0]_{\mathbf{e}} = X^1(\beta_0, \beta_0')$, $[\beta_0']_{\mathbf{e}} = X^2(\beta_0, \beta_0')$.

For the 4-tuple (L_3, L_2, L_1, L_5) there exists a basis $\mathbf{g}$ of L_3 such that two bilinear forms $\beta_1 := [L_3, L_2, L_1]$ and $\beta_1' := [L_3, L_5, L_1]$ are in the standard form, i.e. $[\beta_1]_{\mathbf{g}} = X^1(\beta_1, \beta_1')$, $[\beta_1']_{\mathbf{g}} = X^2(\beta_1, \beta_1')$. Let $\mathbf{e}'$ be a basis of L_1 such that $\omega(\mathbf{g}, \mathbf{e}') = \mathrm{Id}$.

Notice, $[\beta_0]_{\mathbf{e}'} = [\beta_1]_{\mathbf{g}} = X^1(\beta_1, \beta_1')$ in the basis $\mathbf{e}'$. Therefore, we can take matrices of (p,q)-shape transformations $P_{\beta_0\beta_0'}$ and $P_{\beta_1\beta_1'}$ (for more details see Appendix A.5), and define $\mathbf{e_0} := \mathbf{e}P_{\beta_0\beta_0'}$ and $\mathbf{e_1} := \mathbf{e}'P_{\beta_1\beta_1'}$. Then $[\beta_0]_{\mathbf{e_0}} = [\beta_0]_{\mathbf{e_1}} = I_{pq}$ and there exists $U \in \mathrm{O}(p,q)$ such that $\mathbf{e_0} = \mathbf{e_1}U$, where (p,q) is a signature of β_0. We will call this matrix an inner *angle* in the pentagon of Lagrangians $(L_1, L_4, L_2, L_3, L_5)$ (see Figure 6.1).

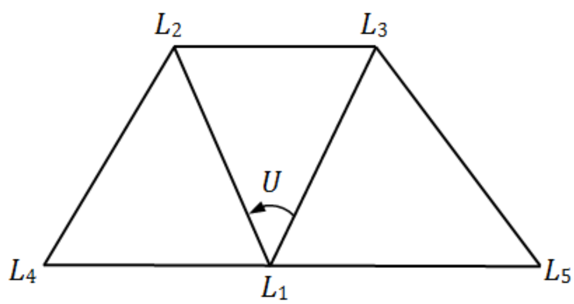

FIGURE 6.1.

Remark 6.6. U is well defined only if the bases $\mathbf{e}, \mathbf{e}'$ of L_1 and $\mathbf{g}$ of L_3 are chosen such that

$$\begin{array}{ll} [\beta_0]_{\mathbf{e}} = X^1(\beta_0, \beta_0') & [\beta_0']_{\mathbf{e}} = X^2(\beta_0, \beta_0') \\ [\beta_1]_{\mathbf{g}} = X^1(\beta_1, \beta_1') & [\beta_1']_{\mathbf{g}} = X^2(\beta_1, \beta_1') \\ \quad\omega(\mathbf{g}, \mathbf{e}') = \mathrm{Id}\,. & \end{array} \tag{6.1}$$

We denote $[L_1, L_5, L_3, L_2, L_4]_{\mathbf{e},\mathbf{e}'} := U$. We denote by $[L_1, L_5, L_3, L_2, L_4]$ the set of all possible $[L_1, L_5, L_3, L_2, L_4]_{\mathbf{e},\mathbf{e}'}$ when $\mathbf{e}, \mathbf{e}'$ satisfy 6.1.

6.2. **Definition of $\mathcal{X}$-coordinates.** Now we can define the general $\mathcal{X}$-coordinates for a triangulated surface $(S, \mathcal{T})$.

Definition 6.7. Let S be a surface with an ideal triangulation $\mathcal{T}$. Let E_{or} be the set of oriented edges of $\mathcal{T}$ and W be the set of angles of $\mathcal{T}$, F be the set of all triangles of $\mathcal{T}$.

A system of $\mathcal{X}$*-coordinates* of rank n on $(S, \mathcal{T})$ is a map

$$x\colon F \sqcup E_{or} \sqcup W \to \{(p,q) \mid p,q \in \mathbb{N} \cup \{0\}, p+q=n\} \sqcup \mathcal{E}(n) \sqcup \bigcup_{p+q=n} O(p,q)$$

such that

- $x(T) \in \{(p,q) \mid p,q \in \mathbb{N} \cup \{0\}, p+q=n\}$. We call $x(T)$ *signature* of the triangle T

- $x(\vec{e}) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}) \in \mathcal{E}(n)$ for each $\vec{e} \in E_{or}$. $x(\vec{e}^{\,-1}) = \sigma(x(\vec{e}))$, where σ is the *edge reorientation map*:
$$\begin{array}{cccc} \sigma & : & \mathcal{E}(n) & \to & \mathcal{E}(n) \\ & & X(b_1, b_2) & \mapsto & X(b_2^*, b_1^*) \end{array}$$
where b_1^*, b_2^* are dual bilinear forms. $\operatorname{sgn}(x(\vec{e})) = x(r(\vec{e}))$, i.e. the signature of $x(\vec{e})$ agree with the signature of the triangle $r(\vec{e})$ which lies to the right form $\vec{e}$;
- $x(w) \in \mathrm{O}(p, q)$ for each $w \in W$, where (p, q) is a signature of the triangle defined as above to which this angle corresponds. $x(w)^{-1} = x(w^{-1})$. For each positive triple of positive angles (w_1, w_2, w_3) it is
$$x(w_3)x(w_2)x(w_1) = \mathrm{Id}$$

We denote by $\mathcal{X}(S, \mathcal{T}, n)$ the set of all $\mathcal{X}$-coordinates of rank n on $(S, \mathcal{T})$.

Remark 6.8. Since we are going to associate triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ to oriented edges, we will write sometimes $x(\vec{e}) = X_{\vec{e}} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}) = X(\beta_1, \beta_2)$ for some pair of forms (β_1, β_2). We will also write $X^i_{\vec{e}}$ for $i \in \{0, 1, 2\}$ for corresponding $X^i(\beta_1, \beta_2)$ because $X^i(\beta_1, \beta_2)$ is completely determined by the triple $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ and the pair (β_1, β_2) is not really important.

Positive $\mathcal{X}$-coordinates are imbedded into the space of general $\mathcal{X}$-coordinates. A coordinate $x \in \mathcal{X}^+(S, \mathcal{T}, n)$ is sent to $x' \in \mathcal{X}(S, \mathcal{T}, n)$ defined by

- $x'(T) = (n, 0)$ for all $T \in F$;
- $x'(e) = (\operatorname{diag} x(e), \varnothing, \varnothing)$ for all $e \in E$;
- $x'(w) = x(w)$ for all $w \in W$.

6.3. **Construction of a decorated representation using $\mathcal{X}$-coordinates.** Let S be a surface with punctures and let $\mathcal{T}$ be an oriented ideal triangulation. Given a decorated representation $[\rho, D] \in \operatorname{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$, we can lift the decoration D to a map $\tilde{D} \colon \tilde{P} \to \operatorname{Lag}(2n, \mathbb{R})$.

Definition 6.9. A system of $\mathcal{X}$-coordinates $x \in \mathcal{X}(S, \mathcal{T}, n)$ is aid to be *admissible* for a representation $[\rho, D] \in \operatorname{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ if

- for each triangle $T = (t_1, t_3, t_2)$ of $\tilde{\mathcal{T}}$, the signature $x(T)$ agrees with the signature of the bilinear form $[\tilde{D}(t_1), \tilde{D}(t_3), \tilde{D}(t_2)]$.
- for each oriented edge $\vec{e} \in \tilde{E}$ on the boundary of the triangles $T = (t_1, t_3, t_2)$ and $T' = (t_2, t_4, t_1)$ of $\tilde{\mathcal{T}}$, the cross ratio $[\tilde{D}(t_1), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$ is conjugated to $-X^0([L_1, L_3, L_2], [L_1, L_4, L_2])^{-1}$;
- for each pentagon in $\tilde{\mathcal{T}}$ as in Figure 6.2, the orthogonal matrix $x(w)$ belongs to the set $[\tilde{D}(t_1), \tilde{D}(t_5), \tilde{D}(t_3), \tilde{D}(t_2), \tilde{D}(t_4)]$.

We now construct as in Section 4.2 a map
$$\operatorname{rep} \colon \mathcal{X}(S, \mathcal{T}, n) \to \operatorname{Hom}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R})),$$

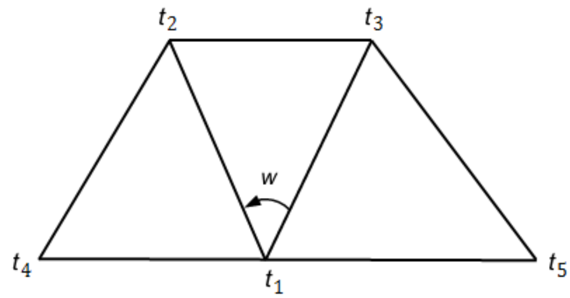

FIGURE 6.2.

such that, for every $x \in \mathcal{X}(S, \mathcal{T}, n)$, $\mathrm{rep}(x)$ is a decorated representation and x is admissible for the representation $\mathrm{rep}(x)$.

For this we let Γ be the graph on the surface introduced in Section 4.2, see Fig. 6.3.

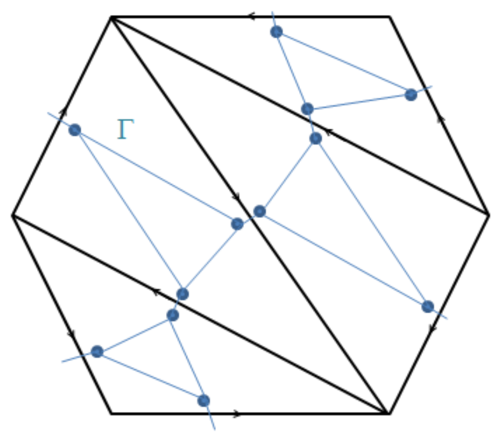

FIGURE 6.3.

To every vertex of Γ we associate an edge coordinate by the rule: let the oriented edge $\vec{r}$ of the triangulation is oriented upwards, then to the point lying to the right from $\vec{r}$ we associate $x(\vec{r})$, to the point lying to the left from $\vec{r}$ we associate $x(\vec{r}^{\,-1})$

We assume that the base point b coincide with one of vertices of Γ. Now, every element $\alpha \in \pi_1(S, b)$ has a representative which is a closed simplicial path in the graph Γ, so

$$\alpha = \alpha_k \circ \cdots \circ \alpha_1,$$

where every α_i is a path along one edge of Γ.

We associate to every α the matrix

$$\rho(\alpha) = A_k \cdots A_1,$$

where A_i is defined as follows:

- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from the right to the left assuming that the edge $\vec{r}$ is oriented upwards, we have
$$E := \begin{pmatrix} 0 & -T^T\Phi \\ T^{-1}\Phi^{-1} & 0 \end{pmatrix}$$
where Φ and T are matrices associated to $x(\vec{r})$ from the definition of the back transformation (see Appendix A.4).
- If α_i is going along an edge of Γ which crosses the oriented edge $\vec{r}$ of the triangulation from the left to the right assuming that the edge $\vec{r}$ is oriented upwards, we have
$$E := -\begin{pmatrix} 0 & -T^T\Phi \\ T^{-1}\Phi^{-1} & 0 \end{pmatrix}$$
where Φ and T are matrices associated to $x(\vec{r}^{\,-1})$ from the definition of the back transformation (see Appendix A.4).
- If α_i is along an edge of Γ that follows the angle w of the triangulation, consider the matrices
$$\hat{U}(X,Y) := \begin{pmatrix} P_Y^T x(w)^T P_X^{-T} & 0 \\ 0 & P_Y^{-1} x(w)^{-1} P_X \end{pmatrix} \tag{6.2}$$
$$T_r(X) = \begin{pmatrix} -\operatorname{Id} & X^1 \\ -X^1 & 0 \end{pmatrix} \quad T_l(X) = (T_r(X))^{-1},$$
where X is the coordinate on the starting vertex of α_i, Y is the coordinate on the ending vertex of α_i, P_X, P_Y are matrices of shape transformations (see Appendix A.5) corresponding to X, resp Y. We have $A_i = \hat{U}(X,Y)T_r(X)$ (resp. $A_i = \hat{U}(X,Y)T_l(X)$) if when going from α_{i-1} to α_i we are turning to the right (resp. to the left). Notice that, $\hat{U}(X,Y)T_r(X) = ((\hat{U}^{-1}(Y,X))T_r(Y))^{-1}$.

After multiplication of all these matrices we get a matrix in $\mathrm{Sp}(2n,\mathbb{R})$ for each curve α. So this process gives us a representation $\rho \in \mathrm{Hom}(\pi_1(S,b), \mathrm{Sp}(2n,\mathbb{R}))$.

This representation admits a natural decoration D. To see this, first, we note that the procedure above works also for non-closed curves.

If b lies in the triangle near to the oriented edge $\vec{r}$ which is adjacent to some puncture and the peripheral curve is just a circle c around this puncture. Then going around c we always are turning either to the right or to the left. Therefore, either $L_{\mathbf{est}}$ or $L_{\mathbf{fst}}$ is preserved by $\rho(c)$ (Figure 6.4). Finally, for each simple peripheral curve γ around some puncture p with start- and endpoint b, we can take a point b' which lies in the triangle adjacent to p. Then we can decompose γ up to homotopy into a path α from b to b', circle c around p and the inverse path α^{-1} from b' to b. For α we get M_α. The matrix corresponding to c preserves some Lagrangian L. Therefore, $\rho(\gamma)$ preserves $M_\alpha^{-1}.L$, and we define $D(\gamma) := M_\alpha^{-1}.L$

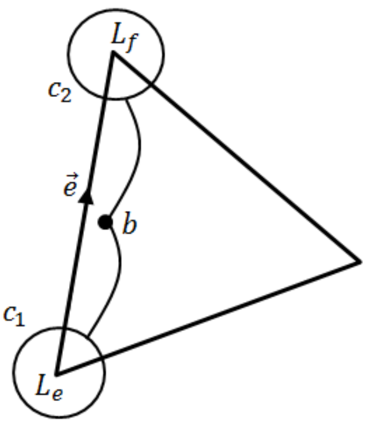

FIGURE 6.4.

For each non-simple peripheral curve which is a power of some simple one, we define a decoration of non-simple peripheral curve to be the decoration of the corresponding simple curve. All other non-simple curves are of the form $\gamma = \beta^{-1}\alpha^n\beta$, where α is simple closed curve, β is some closed curve. So we define $D(\gamma) := \rho(\beta).D(\alpha)$. By construction, this decorated representation is a representative in a standard form of its class. So we define $\mathrm{rep}(x) := (\rho, D)$.

6.4. **The set of $\mathcal{X}$-coordinates associated to a representation.** So far we only constructed a decorated representation given a system of $\mathcal{X}$-coordinates. Now we describe how, given an ideal triangulation, we can associate a system of $\mathcal{X}$-coordinates to a decorated representation $[(\rho, D)]$ so that $[\mathrm{rep}(x)] = [(\rho, D)]$. The procedure described below is very similar to the case of maximal representations. But in this case, one has to be a bit more careful because the cross ratio map is in general not diagonalizable.

We take an ideal triangulation $\mathcal{T}$ of S and choose $b_0 \in S$. Let $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S, b_0), \mathrm{Sp}(2n, \mathbb{R}))$ be a decorated representation.

We lift the oriented triangulation $\mathcal{T}$ of S to the oriented triangulation $\tilde{\mathcal{T}}$ of the universal covering $\tilde{S}$. We also fix a lift $b \in \tilde{S}$ of $b_0 \in S$. Punctures are lifted to visual boundary points of $\tilde{S}$ (after choice of some Riemannian metric of finite area). Using the decoration D, each boundary point can be decorated by a Lagrangian in a unique way. This decoration is $\pi_1(S, b_0)$-equivariant.

We consider the graph Γ associated to this triangulation as in Section 6.3, see Figure 6.3. We can assume that Γ is invariant under the action of $\pi_1(S, b_0)$ on $\tilde{S}$. First, we associate a symplectic basis to each vertex of Γ, a pair (p, q) to each triangle and an element from $\mathcal{E}(n)$ to each oriented edge of the lifted triangulation $\mathcal{T}$. For each vertex b of Γ there is the unique edge r close to which this vertex lies and unique triangle T in which b lies. We take an orientation of the edge $\vec{r}$ such that the vertex b lies to the right from $\vec{r}$. We consider the triangle which is adjacent to T across the edge r. Thus we have a quadrilateral decorated by Lagrangians (L_1, L_3, L_2, L_4). The following symmetric non-degenerate bilinear forms on L_1:

$$\beta_3 := [L_1, L_3, L_2],$$
$$\beta_4 := -[L_1, L_4, L_2]$$

are well-defined.

We put the pair (β_3, β_4) to the standard form, i.e. we choose a basis $\mathbf{e} = (e_1, \ldots, e_n)$ of L_1 such that

$$([\beta_3]_{\mathbf{e}}, [\beta_4]_{\mathbf{e}}) = (X^1(\beta_3, \beta_4), X^2(\beta_3, \beta_4))$$

Since ω identifies L_2 with L_1^*, we define a basis $\mathbf{f}$ of L_2 to be the dual basis to $\mathbf{e}$. So we get in the notation of the previous section:

$$L_1 = \mathrm{Span}(\mathbf{e}) = L_{\mathbf{e}}, \quad L_2 = \mathrm{Span}(\mathbf{f}) = L_{\mathbf{f}}$$
$$L_3 = \mathrm{Span}(\mathbf{e} + \mathbf{f}X^1(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(X^1(\beta_3, \beta_4))$$
$$L_4 = \mathrm{Span}(\mathbf{e} - \mathbf{f}X^2(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(-X^2(\beta_3, \beta_4))$$
$$\omega(e_i, f_j) = \delta_{ij}$$

In this case, we will say that the four tuple (L_1, L_2, L_3, L_4) is in standard position with respect to a symplectic basis $(\mathbf{e}, \mathbf{f})$.

We define the invariants $x(T) := \mathrm{sgn}(\beta_3)$ for the triangle T, $x(\vec{r}) := X(\beta_3, \beta_4)$ for the oriented edge $\vec{r}$ and associate also the symplectic basis $B(b) := (\mathbf{e}, \mathbf{f})$ to the vertex b of Γ.

Because the oriented edge $\vec{r}$ defines the point b uniquely, sometimes we will say that the basis $B(b)$ is associated to the oriented edge $\vec{r}$ and write $B(\vec{r})$.

To define the angle coordinate, we consider a pentagon decorated by Lagrangians as on the Figure 6.5. To each oriented diagonal $\vec{r}_0$ and $\vec{r}_1$ of this pentagon, bases $B(\vec{r}_0) =: (\mathbf{e_0}, \mathbf{f_0})$ of (L_1, L_2) and $B(\vec{r}_1) =: (\mathbf{e_1}, \mathbf{f_1})$ of (L_3, L_1) are associated. So we can define the angle invariant $x(w)$ to be $x(w) := [L_1, L_5, L_3, L_2, L_4]_{\mathbf{e_0},\mathbf{f_1}}$.

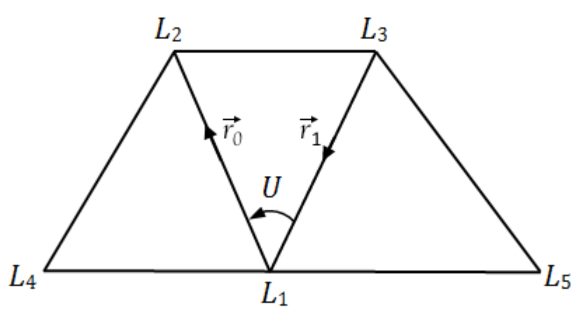

FIGURE 6.5.

Remark 6.10. 1. The choice of bases B is in general not unique. But it can be always chosen in a ρ-equivariant way with respect to the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases because the lifted decoration by Lagrangians is ρ-equivariant. We will always assume that B is ρ-equivariant.

2. By construction, the map x is $\pi_1(S, b_0)$-invariant, therefore, x is well-defined for the triangulation $\mathcal{T}$ of S.

3. By construction, $x(\vec{r}^{\,-1}) = X(\beta_4^*, \beta_3^*) = \sigma(X(\beta_3, \beta_4))$. So our definition of the map x for edges is consistent with the definition of $\mathcal{X}$-coordinates.

4. For each oriented edge $\vec{r}$ of triangulation there are two vertices b_1, b_2 of Γ lying close to $\vec{r}$. In general, there is a lot of possibilities to define $B(b_2)$ if $B(b_1)$ is fixed. We need to fix one of them, which is consistent to the definition of the map rep, namely with the matrix associated to the crossing of an edge. We do the following: Assume $\vec{r}$ is oriented upwards, b_1 lies to the right from $\vec{r}$ and b_2 lies to the left. Let $B(b_1) =: (\mathbf{e}, \mathbf{f})$ then $B(b_2) := (-\mathbf{f}\Phi T, \mathbf{e}\Phi^{-1}T^{-T})$ where Φ and T are matrices associated to $x(\vec{r})$ from the definition of the back transformation (see Appendix A.4).

5. Coordinate which we associate to an edge are in fact connected with the cross ratio operator in the following way:

$$[L_1, L_3, L_2, L_4]_{\mathbf{e}} = [L_4^{-1}]_{\mathbf{f},\mathbf{e}}[L_3]_{\mathbf{e},\mathbf{f}} = -X^0(\beta_3, \beta_4)^{-1}$$

6. This construction does not depend on the choice of a representative (ρ, D) in the class $[\rho, D]$. The triple $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ for each edge is uniquely defined. In contrast, matrices U for each angle are in general not uniquely defined by the representation ρ. To define U, we have chosen a map B which, as we have seen, is in general not unique.

Lemma 6.11. *Let* $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$. *Consider* $x \in \mathcal{X}(S, \mathcal{T}, n)$ *constructed from* $[\rho, D]$ *as above. Then* $[\mathrm{rep}](x) = [\rho, D]$.

Proof. Notice, the bases on vertices of Γ were chosen in compatible way with the construction in the previous section, i.e. let b_1, b_2 be vertices of Γ connected by an edge e. To e the matrix E is associated as in the previous section (going along an angle or crossing an edge of triangulation). Then E maps the basis $B(b_1)$ to $B(b_2)$.

Therefore, by induction, for every loop α based in b, $\mathrm{rep}(\alpha)(B(b)) = B([\alpha]b)$, where by $[\alpha]b$ we understand the action of $[\alpha] \in \pi_1(S, b)$ on vertices of $\Gamma \subseteq \tilde{S}$. But the choice of B is ρ-equivariant, i.e. $\mathrm{rep}(\alpha)(B(b)) = B([\alpha]b) = \rho(\alpha)B(b)$. But the action of $\mathrm{Sp}(2n, \mathbb{R})$ on symplectic bases is exact, therefore, $\mathrm{rep}(\alpha) = \rho(\alpha)$ for all $[\alpha] \in \pi_1(S, b)$, where $\rho(\alpha)$ is written as a matrix with with respect to the basis $B(b)$. $\square$

Corollary 6.12. *The map* $[\mathrm{rep}]$ *is surjective.*

Definition 6.13. Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$, let (ρ, D) be a representative of $[\rho, D]$. Assume, the point b lies in the triangle T_0 near to the upwards oriented edge $\vec{e}$. Assume that peripheral curves α_i (see Figure 6.6), $i \in \{1, 2, 3, 4\}$ are decorated by Lagrangians $L_i \in \mathrm{Lag}(2n, \mathbb{R})$.

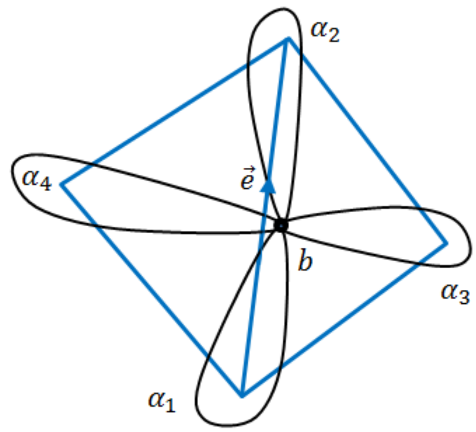

FIGURE 6.6.

We consider bilinear forms β_3, β_4 as above. Then there exists a symplectic basis $(\mathbf{e}, \mathbf{f})$ of $(\mathbb{R}^{2n}, \omega)$ such that

$$L_1 = \mathrm{Span}(\mathbf{e}) = L_{\mathbf{e}}, \quad L_2 = \mathrm{Span}(f) = L_{\mathbf{f}}$$
$$L_3 = \mathrm{Span}(\mathbf{e} + \mathbf{f}X^1(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(X^1(\beta_3, \beta_4))$$
$$L_4 = \mathrm{Span}(\mathbf{e} - \mathbf{f}X^2(\beta_3, \beta_4)) =: L_{\mathbf{e},\mathbf{f}}(-X^2(\beta_3, \beta_4))$$
$$\omega(e_i, f_j) = \delta_{ij}$$

The change-of-basis matrix from the standard basis $(\mathbf{e_{st}}, \mathbf{f_{st}})$ to $(\mathbf{e}, \mathbf{f})$ let be T. Then $(\rho', D') := (T^{-1}\rho T, T^{-1}D) \in [\rho, D]$ is called a *representative in standard form* of $[\rho, D]$. It has the following property:

$$D'(\alpha_1) = L_{\mathbf{e_{st}}}, \; D'(\alpha_2) = L_{\mathbf{f_{st}}},$$
$$D'(\alpha_3) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(X^1(\beta_3, \beta_4)),$$
$$D'(\alpha_4) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(-X^2(\beta_3, \beta_4))$$

Corollary 6.14. *The map* rep *constructed in the previous section gives us for each $x \in \mathcal{X}(S, \mathcal{T}, n)$ a representative in standard form.*

Remark 6.15. Let $(S, \mathcal{T})$ be a surface with ideal triangulation. Assume $b \in S$ lies in the triangle T_0 near to the oriented edge $\vec{e}$. We take four peripheral curves α_i, $i \in \{1, 2, 3, 4\}$ as on the Figure 6.6.

Let $[\rho, D] \in \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ and $x \in \mathcal{X}(S, \mathcal{T}, n)$ is admissible for $[\rho, D]$. Then there exists $(\rho, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ a representative in standard form such that:

$$D(\alpha_1) = L_{\mathbf{e_{st}}}; \;\; D(\alpha_2) = L_{\mathbf{f_{st}}}$$
$$D(\alpha_3) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(X_0^1)$$
$$D(\alpha_4) = L_{\mathbf{e_{st}},\mathbf{f_{st}}}(-X_0^2)$$

where $x(\vec{r}) = X_0$. Moreover, (ρ, D) have the same decoration as $\mathrm{rep}(x)$, and ρ and $\mathrm{rep}(x)$ act in the same way on $D(\pi_1^{per}(S, b))$.

Remark 6.16. Let $x \in \mathcal{X}(S, \mathcal{T}, n)$ be admissible for $[\rho_1, D_1]$ and for $[\rho_2, D_2]$. Then there exist $(\rho_1, D_1) \in [\rho_1, D_1]$ and $(\rho_2, D_2) \in [\rho_2, D_2]$ representatives in a standard form such that $D_1 = D_2$. In particular, the decoration of $\operatorname{rep}(x)$ coincides up to $\operatorname{Sp}(2n, \mathbb{R})$-action with decoration of each decorated representation for which x is admissible.

Remark 6.17. If $x \in \mathcal{X}(S, \mathcal{T}, n)$ is admissible $\mathcal{X}$-coordinates for $[\rho, D] \in \operatorname{Rep}^d_{\mathcal{T}}(\pi_1(S, b), \operatorname{Sp}(2n, \mathbb{R}))$, then in general it is wrong that $[\operatorname{rep}(x)] = [\rho, D]$.

As we have seen, angle coordinates are not uniquely defined. Sometimes different collections of angle coordinates define the same representation. Now we are going to find out how the angles can be changed so that the representation stays the same.

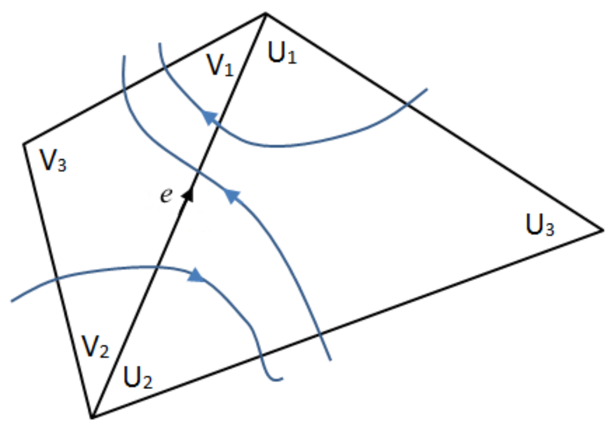

FIGURE 6.7.

We take two adjacent by an edge e triangles. The coordinate on the edge is X_e (oriented as on fig. 6.7). The coordinate associated to the opposite orientation of e we denote by $\tilde{X}_e$. Signature of right triangle assume to be $(p, q) = \operatorname{sgn}(X_e^1) = \operatorname{sgn}(\tilde{X}_e^2)$, signature of left triangle assume to be $(p', q') = \operatorname{sgn}(X_e^2) = \operatorname{sgn}(\tilde{X}_e^1)$. We also assume that all angles are oriented counterclockwise with respect to the triangle.

Theorem 6.18. *Let $x \in \mathcal{X}(S, \mathcal{T}, n)$. Let us change angle coordinates along the edge e in a following way:*

$$U_1' = WU_1,\ V_1' = V_1 W'^{-1},\ U_2' = U_2 W^{-1},\ V_2' = W' V_2 \tag{6.3}$$

where

$$W \in \operatorname{O}(p, q) \cap \operatorname{O}(P_{X_e}^{-T} X_e^2 P_{X_e}^{-1})$$
$$W' := D^{-1} W^T D$$
$$D := P_{X_e}^{-T} \Phi_{X_e} T_{X_e} P_{\tilde{X}_e}^{-1}$$

This gives us another $x' \in \mathcal{X}(S, \mathcal{T}, n)$. Then $[\operatorname{rep}(x)] = [\operatorname{rep}(x')]$.

Proof. First, we need the following proposition:

Proposition 6.19.

$$W' \in O(p', q') \cap O(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1})$$

Proof. First, we note that $D^{-T} I_{p'q'} D^{-1} = P_{X_e}(X_e^2)^{-1} P_{X_e}^T$:

$$\begin{aligned}D^{-T} I_{p'q'} &= P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} P_{\tilde{X}_e}^T I_{p'q'} = P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} \tilde{X}_e^1 P_{\tilde{X}_e}^{-1} = \\ &= [(X^2)^{-1}\Phi T = \Phi^{-1} T^{-T} \tilde{X}^1] = P_{X_e}(X_e^2)^{-1} \Phi_{X_e} T_{X_e} P_{\tilde{X}_e}^{-1} = \\ &= P_{X_e}(X_e^2)^{-1} P_{X_e}^T D\end{aligned}$$

Therefore,

$$\begin{aligned}W'^T I_{p'q'} W' &= D^T W D^{-T} I_{p'q'} D^{-1} W^T D = D^T W P_{X_e}(X_e^2)^{-1} P_{X_e}^T W^T D = \\ &= [W \in O(P_{X_e}^{-T} X_e^2 P_{X_e}^{-1})] = D^T P_{X_e}(X_e^2)^{-1} P_{X_e}^T D = I_{p'q'}\end{aligned}$$

So $W' \in O(p', q')$.

Second, we note that $D^{-T} P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} D^{-1} = I_{pq}$:

$$\begin{aligned}D^{-T}(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}) &= P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} P_{\tilde{X}_e}^T P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} = P_{X_e} \Phi_{X_e}^{-1} T_{X_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1} = \\ &= [X^1 \Phi T = \Phi^{-1} T^{-T} \tilde{X}^2] = P_{X_e} X_e^1 \Phi_{X_e} T_{X_e} P_{\tilde{X}_e}^{-1} = [P_{X_e}^T I_{pq} P_{X_e} = X_e^1] = I_{pq} D\end{aligned}$$

Therefore,

$$\begin{aligned}W'^T(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}) W' &= D^T W D^{-T}(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}) D^{-1} W^T D = \\ &= D^T W^T I_{pq} W D = [W \in O(p,q)] = D^T I_{pq} D = P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1}\end{aligned}$$

So $W' \in O(P_{\tilde{X}_e}^{-T} \tilde{X}_e^2 P_{\tilde{X}_e}^{-1})$.

□

Using the last proposition, it is easy to calculate that:

$$\hat{V}_1 T_r E_{X_e} \hat{U}_1 = \hat{V}_1' T_r E_{X_e} \hat{U}_1'$$

$$\hat{U}_2 T_r E_{X_e} \hat{V}_2 = \hat{U}_2' T_r E_{X_e} \hat{V}_2'$$

$$\hat{V}_1 T_r E_{\tilde{X}_e} \hat{U}_2^{-1} = \hat{V}_1' T_r E_{\tilde{X}_e} \hat{U}_1'$$

So holonomies of all curves are not changed. □

Corollary 6.20. *Let $x, x' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $\operatorname{rep}(x) = \operatorname{rep}(x')$. Let $w \in W$ be an angle which is adjacent to some oriented edge $\vec{e}$. Then angle coordinates of x along $\vec{e}$ can be changed as above to coordinates $x'' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $x'(w) = x''(w)$ and $x(w') = x''(w')$ for all angles w' which are not adjacent to $\vec{e}$*

Lemma 6.21. *The only possible changes of angle coordinates so that the reconstructed representation does not change are given by formulas 6.3.*

Proof. (Sketch) We take the surface S of genus g and k punctures and fix the triangulation and the base point as on the picture. For another choice of triangulation the proof is similar.

We take $x, x' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $\mathrm{rep}(x) = \mathrm{rep}(x')$. We assume that x, x' define two different collections of angles $\{U_i\}$ and $\{U_i'\}$. Now we show that by correction of angles $\{U_i\}$ by formulas above we can get the collection $\{U_i'\}$.

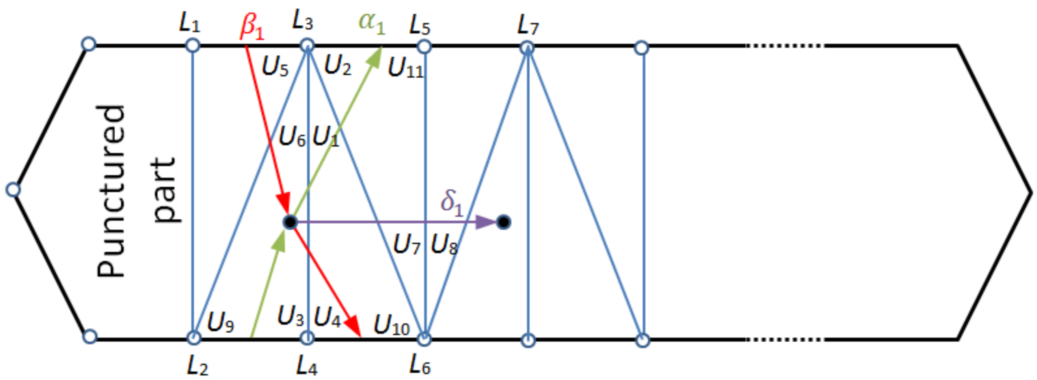

FIGURE 6.8.

Using Corollary 6.20 we correct all upper angles ($U_5, U_6, U_1, U_2, U_{11}, \dots$ see Figure 6.8) getting $x'' \in \mathcal{X}(S, \mathcal{T}, n)$ such that $\mathrm{rep}(x) = \mathrm{rep}(x') = \mathrm{rep}(x'')$. Note, that the number of these corrected angles agree with the total number of (non-oriented edges) since we correct each angle along exactly one edge. It makes automatically that some other angles agree ($U_7, U_8, \dots$) because product of angles in one triangle is always Id. To see that all other agree, it is enough to look at generators of $\pi_1(S, b)$ ($\alpha_1, \beta_1, \dots$ on Figure 6.8). Since their holonomies agree for $\mathrm{rep}(x) = \mathrm{rep}(x') = \mathrm{rep}(x'')$, all other angles ($U_9, U_3, U_4, U_{10}, \dots$) agree automatically. So we get $x = x''$

□

Remark 6.22 (Changing of $\mathcal{X}$-coordinates by a flip). As we have see already for positive $\mathcal{X}$-coordinates, it is difficult to write an explicit formula of change of coordinates by a flip. In the general case, it is even more difficult. However we will see in Section 8.4, using $\mathcal{A}$-coordinates we can give explicit formulas, which tell us how the $\mathcal{X}$-coordinates change.

6.5. **"Degenerate representations" of constant signature.** In this section we describe very explicitly the coordinates for degenerate representations of constant signature, these correspond to representation that factor through embeddings

$$\mathrm{SL}(2, \mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(p, q) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$$

$$\mathrm{PSL}(2,\mathbb{K}) \times \mathrm{PO}(p,q) \hookrightarrow \mathrm{PSp}(2n,\mathbb{R})$$

where $p+q=n$, $\mathbb{Z}_2 = \{1,-1\}$ is the group with two elements considered as a multiplicative subgroup of $\mathbb{R}^*$. $\mathbb{Z}_2$ can be embedded into the center of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{O}(p,q)$ diagonally , so the tensor product is well-defined.

If we take a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

then

$$\hat{A} = \begin{pmatrix} a\,\mathrm{Id}_n & bI_{pq} \\ cI_{pq} & d\,\mathrm{Id}_n \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$$

Also if we take some matrix $U \in O(p,q)$ then

$$\hat{U} = \begin{pmatrix} U^{-T} & 0 \\ 0 & U \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$$

The maps $\hat{}$ are in both cases injective homomorphisms. Because $I_{pq}U = U^{-T}I_{pq}$, matrixes $\hat{A}$ and $\hat{U}$ commute. That give us an homomorphism

$$\phi\colon\ \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(p,q) \to \mathrm{Sp}(2n,\mathbb{R})$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, U\right) \mapsto \hat{A}\hat{U} = \begin{pmatrix} aU^{-T} & bI_{pq}U \\ cI_{pq}U^{-T} & dU \end{pmatrix}$$

$\mathrm{Ker}(\phi) = \{\pm(\mathrm{Id},\mathrm{Id})\} \cong \mathbb{Z}_2$. So we get embeddings

$$\mathrm{SL}(2,\mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(p,q) \hookrightarrow \mathrm{Sp}(2n,\mathbb{R})$$

$$\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PO}(p,q) \hookrightarrow \mathrm{PSp}(2n,\mathbb{R})$$

We will identify pair $(A,U) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(p,q)$ up to common sign and $\hat{A}\hat{U} \in \mathrm{Sp}(2n,\mathbb{R})$ and also pair $(A,U) \in \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PO}(p,q)$ and $\hat{A}\hat{U} \in \mathrm{PSp}(2n,\mathbb{R})$.

Now we take some surface S with punctures and an ideal triangulation and a base point $b \in S$ as above.

Lemma 6.23. *Let $x \in \mathcal{X}(S,\mathcal{T},n)$ be given. We assume that for x all X_e^0 are scalar matrices for all edges e. Then the reconstructed representation*

$$\mathrm{rep}(x) = \rho\colon \pi_1(S,b) \to \mathrm{SL}(2,\mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(p,q)$$

Moreover, $\rho = \rho_1 \otimes_{\mathbb{Z}_2} \rho_2$, where for each loop γ, $\rho_2(\gamma)$ is the product of corresponding to γ angle coordinates, $\rho_1\colon \pi_1(S,b) \to \mathrm{SL}(2,\mathbb{R})$.

Proof. $\rho(\gamma)$ is a product of matrices as E_{X_e}, T_r, T_l, $\hat{U}$, where $X_e^0{=}l_e\,\mathrm{Id}$ for some $l_e \neq 0$. Therefore,

$$E_{X_e} = \phi\left(\begin{pmatrix} 0 & -\sqrt{|l_e|} \\ \sqrt{|l_e|^{-1}} & 0 \end{pmatrix}, \mathrm{Id}\right)$$

$$T_r = \phi\left(\begin{pmatrix} -1 & \mathrm{sgn}(l_e) \\ -\mathrm{sgn}(l_e) & 0 \end{pmatrix}, \mathrm{Id}\right)$$

$$T_l = \phi\left(\begin{pmatrix} 0 & -\operatorname{sgn}(l_e) \\ \operatorname{sgn}(l_e) & -1 \end{pmatrix}, \mathrm{Id}\right)$$

$$\hat{U} = \phi(\mathrm{Id}, U)$$

Since $\hat{U}$ commutes with E_Φ, T_r, T_l, we have

$$\rho(\gamma) \in \mathrm{SL}(2,\mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(p,q)$$

and $\rho_2(\gamma) = \prod_{i=1}^k U_i$ is a product of angle coordinates. $\square$

Corollary 6.24. *Let a collection of coordinates on the triangulate surface S be given. We assume that X_e^0 are scalar matrices for all edges e. Then the reconstructed representation*

$$\rho\colon \pi_1(S,b) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PO}(p,q)$$

Moreover, for each curve γ $\rho_2(\gamma) = Pr_2(\rho(\gamma))$ is a product of corresponding to this curve angle coordinates.

For the lemma 6.23 it is possible to prove some "incomplete" converse lemma:

Lemma 6.25. *Let*

$$\rho\colon \pi_1(S,b) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PO}(p,q)$$

and there exist a decoration D such that

$$(\rho_1, D) := (Pr_1(\rho), D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi(S,b), \mathrm{PSL}(2,\mathbb{R}))$$

Then $D' := D \otimes \mathbb{R}^n$ is a decoration for ρ. Moreover, for all $x \in \mathcal{X}(S,\mathcal{T},n)$ such that $[\mathrm{rep}(x)] = [\rho, D']$ edge coordinates X_e^0 for all edges e are scalar matrices.

Proof. For some peripheral curve γ let $v := D(\gamma)$ be an eigenvector of $\rho_1(\gamma)$. Then by definition of action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PO}(p,q)$ on $\mathbb{R}^2 \otimes \mathbb{R}^n$ Lagrangian $\mathrm{Span}(v) \otimes \mathbb{R}^n$ is an eigenspace of $\rho(\gamma)$. So D' is well-defined.

If we have four Lagrangians of the form $V_i = \mathrm{Span}(v_i) \otimes \mathbb{R}^n$, $i = 1, \ldots, 4$, then the corresponding cross ratio map is:

$$[V_1, V_2, V_3, V_4] = [v_1, v_2, v_3, v_4]\,\mathrm{Id}_{V_1},$$

which matrix is scalar in each basis of V_1. So for corresponding e we have $X_e^0 = [v_1, v_2, v_3, v_4]\,\mathrm{Id}$. $\square$

Corollary 6.26. *Let*

$$\rho\colon \pi_1(S,b) \to \mathrm{SL}(2,\mathbb{R}) \otimes_{\mathbb{Z}_2} \mathrm{O}(p,q)$$

be a representation, $\rho = \rho_1 \otimes_{\mathbb{Z}_2} \rho_2$, where

$$(\rho_1, D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi(S,b), \mathrm{SL}(2,\mathbb{R}))$$

for some decoration D. Then $D' := D \otimes \mathbb{R}^n$ is a decoration for ρ and for all $x \in \mathcal{X}(S,\mathcal{T},n)$ such that $[\mathrm{rep}(x)] = [\rho, D']$ edge coordinates X_e^0 for all edges e are scalar matrices.

Remark 6.27. This lemma shows that edge coordinates "know" nothing about the second $\mathrm{O}(p,q)$-part of representation. For this part angle coordinates are responsible.

Remark 6.28. Remember that since $\pi_1(S)$ is free with $2g+k-1$ generators, the following spaces are homeomorphic:

$$\mathrm{Hom}(\pi_1(S), \mathrm{O}(p,q)) \cong \mathrm{O}(p,q)^{2g+k-1}$$
$$\mathrm{Rep}(\pi_1(S), \mathrm{O}(p,q)) \cong \mathrm{O}(p,q)^{2g+k-1} / \mathrm{O}(p,q)$$

Corollary 6.29. *The following spaces are homeomorphic:*

$$\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \cong \mathrm{Rep}(\pi_1(S,b), \mathrm{O}(n)) \cong \mathrm{O}(n)^{2g+k-1} / \mathrm{O}(n)$$

where $\mathcal{D}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ *is the subspace of* $\mathcal{M}^d_{\mathcal{T}}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ *of all degenerate maximal representations with edge coordinates equal to* $(1,\dots,1)$.

7. $\mathcal{X}$-coordinates for representations into central extensions

We introduced $\mathcal{X}$-coordinates for decorated representations into $\mathrm{Sp}(2n,\mathbb{R})$ using invariants of Lagrangian subspaces. As we remarked before, the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$ is not effective, but factors through $\mathrm{PSp}(2n,\mathbb{R})$. Therefore, the construction of $\mathcal{X}$-coordinates works as well for decorated representations into $\mathrm{PSp}(2n,\mathbb{R})$,The notions of decoration and transversality are well-defined because the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$ is just the lift of the action of $\mathrm{PSp}(2n,\mathbb{R})$ on $\mathrm{Lag}(2n,\mathbb{R})$. We only have to modify the angle invariants, as they now take values in $\mathrm{PO}(p,q)$.

We can then similarly define a map rep from $\mathcal{X}$-coordinates to the space of transverse decorated representations $\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S,b), \mathrm{PSp}(2n,\mathbb{R}))$.

Note that $\mathrm{Sp}(2n,\mathbb{R})$ is a central extension of $\mathrm{PSp}(2n,\mathbb{R})$ by the abelian group $\mathbb{Z}_2$. In this sections we extend the construction of $\mathcal{X}$-coordinates to representations into arbitrary central extensions of $\mathrm{PSp}(2n,\mathbb{R})$. The most interesting cases are the connected coverings of $\mathrm{PSp}(2n,\mathbb{R})$ are classified by the subgroups of $\mathbb{Z}$, e.g. $\mathrm{Sp}(2n,\mathbb{R})$ corresponds to $2\mathbb{Z}$, the metaplectic group to $4\mathbb{Z}$, the universal covering to (0).

Let G be a central extension of $\mathrm{PSp}(2n,\mathbb{R})$ by an abelian group A that is determined by a cocycle $c \in H^2(\mathrm{PSp}(2n,\mathbb{R}), A)$. We fix a bijection $\theta\colon \mathrm{PSp}(2n,\mathbb{R}) \times A \to G$. We denote the restriction of θ on the subset $\mathrm{PSp}(2n,\mathbb{R}) \times \{0\} \cong \mathrm{PSp}(2n,\mathbb{R})$ by ι as above.

Let S be a surface with punctures as above, $\mathcal{T}$ be an ideal triangulation of S. Each representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$ projects to some representation $\rho' \in \mathrm{Hom}(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$. Assume ρ' admits a decoration D which is transverse with respect to $\mathcal{T}$. If D is fixed, then $(\rho', D) \in \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), \mathrm{PSp}(2n,\mathbb{R}))$.

Definition 7.1. The pair (ρ, D) constructed as above is called *decorated representation into the central extension G transverse with respect to $\mathcal{T}$*. The set of all decorated representation into the central extension G transverse with respect to $\mathcal{T}$ is denoted by $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), G)$.

Definition 7.2. We denote

$$\mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), G) := \mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), G)/G$$

Definition 7.3. A representation $\rho \in \mathrm{Hom}(\pi_1(S), G)$ is called *maximal* if it projects to a maximal representation $\rho' \in \mathrm{Hom}_{max}(\pi_1(S), \mathrm{PSp}(2n, \mathbb{R}))$. The space of all maximal representations into G is denoted by $\mathrm{Hom}_{max}(\pi_1(S), G)$. The space of all maximal decorated representations into G is denoted by $\mathrm{Hom}^d_{max}(\pi_1(S), G)$.

Definition 7.4. We denote

$$\mathcal{M}(\pi_1(S), G) := \mathrm{Hom}_{max}(\pi_1(S), G)/G$$

$$\mathcal{M}^d(\pi_1(S), G) := \mathrm{Hom}^d_{max}(\pi_1(S), G)/G$$

Consider the embedding:

$$\begin{array}{rccc} \psi\colon & \mathrm{PO}(p,q) & \hookrightarrow & \mathrm{PSp}(2,\mathbb{R}) \\ & U & \mapsto & \mathrm{diag}(U, U^{-T}) \end{array}$$

Then there exists the unique subgroup $G(p,q) < G$ such that $A < G(p,q)$ and projects to $\psi(\mathrm{PO}(p,q))$.

Before we give the definition of $\mathcal{X}$-coordinates for central extension, we recall that $\mathcal{E}(n)$ is the set of all triples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$ where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}$ are of the form as in the Theorem 6.1 with

$$\dim \mathcal{J}_1 + \dim \mathcal{J}_2 + \dim \mathcal{K} = n.$$

Definition 7.5 ($\mathcal{X}$-coordinates for central extension)**.** Let S be a surface with an ideal triangulation $\mathcal{T}$. Let E_{or} be the set of oriented edges of $\mathcal{T}$ and W be the set of angles of $\mathcal{T}$, F be the set of triangles of $\mathcal{T}$.

A system of $\mathcal{X}$*-coordinates* of rank n for the central extension G with respect to $\mathcal{T}$ is a map

$$x\colon F \sqcup E_{or} \sqcup W \to \{(p,q) \mid p,q \in \mathbb{N} \cup \{0\}, p+q=n\} \sqcup \mathcal{E}(n) \sqcup \bigcup_{p+q=n} G(p,q)$$

such that

- the triangle invariant $x(T)$ takes values in $\{(p,q) \mid p,q \in \mathbb{N} \cup \{0\}, p+q=n\}$. We call $x(T)$ also *signature* of the triangle T
- the edge invariant $x(\vec{e})$ is given by $x(\vec{e}) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}) \in \mathcal{E}(n)$ for each $\vec{e} \in E_{or}$. $\mathcal{X}(\vec{e}^{-1}) = \sigma(\mathcal{X}(\vec{e}))$, where σ is the *edge reorientation map*:

$$\begin{array}{cccc} \sigma & : & \mathcal{E}(n) & \to & \mathcal{E}(n) \\ & & X(b_1,b_2) & \mapsto & X(b_2^*, b_1^*) \end{array}$$

  where b_1^*, b_2^* are dual bilinear forms to b_1, b_2. $\mathrm{sgn}(x(\vec{e})) = x(r(\vec{e}))$, i.e. the signature of $x(\vec{e})$ agree with the signature of the triangle $r(\vec{e})$ which lies to the right form $\vec{e}$;

- the angle invariant $x(w)$ takes values in $G(p,q)$ for each $w \in W$, where (p,q) is a signature of the triangle defined as above which this angle corresponds to. $U(w^{-1}) = U(w)^{-1}$. For each positive triple of positive angles (w_1, w_2, w_3) is subject to the condition
$$U(w_3)U(w_2)U(w_1) = \mathrm{Id}\,.$$

We denote by $\mathcal{X}_G(S, \mathcal{T}, n)$ the set of all $\mathcal{X}$-coordinates of rank n for the central extension G on $(S, \mathcal{T})$.

By the same procedure as for $\mathcal{X}$-coordinates for $\mathrm{Sp}(2n,\mathbb{R})$, see Section 6.3, we can construct a map rep_G from the space of $\mathcal{X}$-coordinates to the space of decorated homomorphism $\mathrm{Hom}^d_{\mathcal{T}}(\pi_1(S), G)$, which induces a surjective map
$$[\mathrm{rep}_G] \colon \mathcal{X}_G(S, \mathcal{T}, n) \to \mathrm{Rep}^d_{\mathcal{T}}(\pi_1(S), G).$$

Using the map $[\mathrm{rep}_G]$ restricted to the positive locus of $\mathcal{X}_G(S, \mathcal{T}, n)$, i.e. the subset of $\mathcal{X}_G(S, \mathcal{T}, n)$ such that all triangle invariants are $(n, 0)$, as in the Section 5.2.1, we can study the homotopy type of $\mathcal{M}^d(\pi_1(S), G)$. Namely, we can get the following result:

Theorem 7.6. *The space of decorated maximal representations $\mathcal{M}^d(\pi_1(S), G)$ is homotopically equivalent to $G(n,0)^{2g+k-1}/G(n,0)$, where g is the genus of S, k is the number of punctures and the quotient is taken by the action of $G(n,0)$ on $G(n,0)^{2g+k-1}$ by simultaneous conjugation.*

8. $\mathcal{A}$-coordinates for framed representations

In this section we introduce $\mathcal{A}$-coordinates associated to framed representations of $\pi_1(S)$. The $\mathcal{A}$-coordinates endow the space of framed representations with a structure of non-commutative cluster algebra, as defined by Berenstein–Retakh [2]. Every triangulation gives a set of cluster functions into the non-commutative algebra of $n \times n$ matrices, and when an edge of the triangulation is flipped, the cluster functions change with the formulae given in [2]. In this way we give a geometric interpretation to the algebraic theory developed there. We also explain the relation between $\mathcal{A}$-coordinates and $\mathcal{X}$-coordinates and give precise formulas for the flip in both coordinates. Before we focus on framed representations we just consider framed triangulations.

Remark 8.1. In this section, it is not essential at all that we are working over real numbers. We can replace the field $\mathbb{R}$ by any other field. But to be consistent with the previous sections, we will provide the construction for the group $\mathrm{Sp}(2n,\mathbb{R})$.

8.1. **Oriented framed triangulations and their representations.**

Let $C \subset \mathbb{R}^2$ be a disc, and consider an oriented triangulation $\mathcal{T}$ of C, i.e. an embedded oriented graph $\mathcal{T} = (W, E)$ such that $C \smallsetminus \bigcup E$ is a union of triangles. For each edge e an orientation is fixed, i.e. we define a map $e \colon \{0,1\} \to W$ such that $\{e(0), e(1)\} = \partial e$.

Definition 8.2. A triangulation is called *framed decorated* if

- for each vertex $w \in W$ a framed n-dimensional subspace $(L(w), v(w))$ of $\mathbb{R}^{2n}$ is fixed
- $L(e(0))$ and $L(e(1))$ are transverse for each edge $e \in E$ for each edge e.

We denote $v(e) := (v(e(0)), v(e(1)))$.

For each edge $e \in E$ we take the midpoint $p(e)$ of e and then consider the following oriented graph $\Gamma = (P, R)$: the set of vertices is $P = \{p(e) \mid e \in E\}$, two vertices $p(e)$ and $p(e')$ are connected by an edge from R if and only if e and e' belong to the same triangle of $\mathcal{T}$. We also fix some orientation of edges of Γ. So we have a basis $v(p(e)) := v(e)$ for each vertex from P

A path in Γ is a sequence of vertices so that each two adjacent vertices are connected by an edge. We denote by $\mathcal{P}(\Gamma)$ the set of all paths of Γ. Our goal is to construct a map $\rho\colon \mathcal{P}(\Gamma) \to \mathrm{GL}(2n, \mathbb{R})$ such that for each path $\gamma = p(e_1) \dots p(e_k)$ holds:

$$\rho(\gamma)v(e_1) = v(e_k)$$

By this property this map is uniquely defined.

To do this, we define a labeling of each edge of Γ by $\mathrm{GL}(2n, \mathbb{R})$ matrices $A(r)$ for all $r \in R$, where $A(r)$ is the change of basis matrix from the basis $v(r(0))$ to $v(r(1))$. And define:

$$\rho(\gamma) := A(r_1)^{\varepsilon_1} \dots A(r_k)^{\varepsilon_k}$$

where $\varepsilon_i = 1$ if we go along r_i with respect to its orientation, $\varepsilon_i = -1$ otherwise. By construction, this map has the necessary property.

Corollary 8.3. *Let a triangulation $\mathcal{T}$ with corresponding graph $\Gamma = (P, R)$ be given. We fix $p_0 = p(e_0) \in P$ and assume that $v(e_0(0))$, $v(e_0(1))$ and ρ are given. Then we can reconstruct the whole oriented framed decorated triangulation.*

Remark 8.4. Let $(\mathbb{R}^{2n}, \omega)$ be a symplectic vector space. For each path $\gamma = p(e_0) \dots p(e_k)$ such that $\omega(v(e_k(0)), v(e_k(1))) = \omega(v(e_0(0)), v(e_0(1)))$ the matrix $\rho(\gamma) \in \mathrm{Sp}(2n, \mathbb{R})$.

8.2. **Oriented framed triangulations and $\mathcal{A}$-coordinates.** In this section we assume that a standard symplectic form ω on $\mathbb{R}^{2n}$ is given. Consider a framed by Lagrangians decorated triangulation. We can define a symplectic Λ-length associated to each edge using the formula:

$$\Lambda_e = \omega(v(e(0)), v(e(1)))$$

By given framed decorated triangulation the symplectic Λ-length for each edge is uniquely defined.

Definition 8.5. Let $\mathcal{T}$ be a triangulation. The map

$$a\colon E \to \mathrm{GL}(n, \mathbb{R})$$

is called *$\mathcal{A}$-coordinates of rank n* if it satisfies the triangle relation for symplectic Λ-lengths (see Section 2.5).

Lemma 8.6. *The oriented triangulation with associated $\mathcal{A}$-coordinates defines an oriented framed decorated triangulation uniquely up to action of* $\mathrm{Sp}(2n,\mathbb{R})$ *on* $\mathrm{Lag}^{fr}(2n,\mathbb{R})$.

Proof. First, we reconstruct the map $\rho\colon \mathcal{P}(\Gamma)\to \mathrm{Sp}(2n,\mathbb{R})$. To do this, we have to construct A_r for all $r\in R$. Without restriction of generality, we can assume that r goes from $p(e)$ to $p(e')$ so that $e(0)=e'(0)$. We denote:

$$w_0 := e(0)=e'(0),\ w:=e(1),\ w':=e'(1)$$

Let e'' be an edge connecting w and w'. Without restriction of generality, assume e'' goes from w to w'.

Because of equality

$$(v(w_0),v(w'))=(v(w_0),v(w))A_r$$

we can see that

$$A_r=\begin{pmatrix}\mathrm{Id} & X\\ 0 & Y\end{pmatrix}$$

and $v(w')=v(w_0)X+v(w)Y$. But

$$c_e=\omega(v(w_0),v(w)),\ c_{e'}=\omega(v(w_0),v(w')),\ c_{e''}=\omega(v(w),v(w'))$$

therefore,

$$c_{e'}=\omega(v(w_0),v(w'))=\omega(v(w_0),v(w_0)X+v(w)Y)= \\ =\omega(v(w_0),v(w))Y=c_eY$$

$$c_{e''}=\omega(v(w),v(w'))=\omega(v(w),v(w_0)X+v(w)Y)= \\ =\omega(v(w),v(w_0))X=-c_e^TX$$

So we can calculate X and Y:

$$X=-c_e^{-T}c_{e''},\ Y=c_e^{-1}c_{e'}$$

$$A_s=\begin{pmatrix}\mathrm{Id} & -c_e^{-T}c_{e''}\\ 0 & c_e^{-1}c_{e'}\end{pmatrix}$$

and reconstruct ρ.

In the last step, we fix some $p_0=p(e_0)\in P$ and try to reconstruct $v(e_0(0))$, $v(e_0(1))$ up to action of $\mathrm{Sp}(2n,\mathbb{R})$. Since $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on $\mathrm{Lag}^{fr}(2n,\mathbb{R})$, we can choose as $v(e_0(0))$ the first n vectors of the standard symplectic basis of $(\mathbb{R}^{2n},\omega)$. They span some Lagrangian subspace. Since $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on pairs of transverse Lagrangians, we can choose as $L(e_0(1))$ the span of the last n basis vectors of the standard symplectic basis of $(\mathbb{R}^{2n},\omega)$. We denote by $\tilde v(e_0(1))$ the last n basis vectors of the standard basis of $\mathbb{R}^{2n}$.

Because $v(e_0(1))$ spans $L(e_0(1))$, it is $v(e_0(1))=\tilde v(e_0(1))Z$. Since

$$c_{e_0}=\omega(v(e_0(0)),v(e_0(1)))=\omega(v(e_0(0)),\tilde v(e_0(1))Z)=Z$$

we get a basis $v(e_0(1))=\tilde v(e_0(1))c_{e_0}$. By Corollary 8.3 the oriented framed decorated triangulation can be reconstructed. □

Remark 8.7. For oriented triangulations with $\mathcal{A}$-coordinates, all properties of symplectic Λ-lengths as the exchange relation, the triangle relation hold.

8.3. **Framed representations and $\mathcal{A}$-coordinates.**

Definition 8.8. Let $(S, \mathcal{T})$ be a punctured surface with an oriented ideal triangulation. The map

$$a\colon E \to \mathrm{GL}(n, \mathbb{R})$$

is called *$\mathcal{A}$-coordinates of rank n* if it satisfies the triangle relation for symplectic Λ-lengths (see Section 2.5). The set of all $\mathcal{A}$-coordinates on $(S, \mathcal{T})$ is denoted by $\mathcal{A}(S, \mathcal{T}, n)$.

Remark 8.9. Because of the triangle relation the set $\mathcal{A}(S, \mathcal{T}, n)$ for $n > 1$ is not isomorphic to $\mathrm{GL}(n, \mathbb{R})^{\#E}$. It is a closed subset of $\mathrm{GL}(n, \mathbb{R})^{\#E}$ of positive codimension.

Note that in the case of framed representations, the situation is considerably simpler than for decorated representations. We have

Theorem 8.10. *There is a 1-1 correspondence between $\mathcal{A}(S, \mathcal{T}, n)$ and $X^{fr}_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$.*

Proof. Let $(\rho, D) \in \mathrm{Hom}^{fr}_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$. We can lift the oriented triangulation $\mathcal{T}$ together with the decoration D to the universal covering of S. This gives us a ρ-equivariant oriented framed decorated triangulation $\tilde{\mathcal{T}}$ on $\mathbb{H}^2$. By this triangulation $\tilde{\mathcal{T}}$ a $\pi_1(S, b)$-invariant collection of $\mathcal{A}$-coordinates on $\tilde{\mathcal{T}}$ can be defined. Therefore, this collection of $\mathcal{A}$-coordinates is well-defined on the oriented triangulation $\mathcal{T}$ of S.

If we have a collection of $\mathcal{A}$-coordinates on an oriented triangulation $\mathcal{T}$ of S, we can lift it to the $\pi_1(S, b)$-invariant collection of $\mathcal{A}$-coordinates on $\tilde{\mathcal{T}}$ on $\mathbb{H}^2$. We can reconstruct $(\rho', D') \in \mathrm{Hom}^{fr}_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ using the procedure described in Section 8.2. If (ρ', D') was reconstructed using $\mathcal{A}$-coordinated which was got by the representation (ρ, D), then $[\rho', D'] = [\rho, D] \in X^{fr}_{\mathcal{T}}(\pi_1(S, b), \mathrm{Sp}(2n, \mathbb{R}))$ because up to symplectic change-of-basis they act in the same on a fixed symplectic basis. $\square$

8.4. **$\mathcal{X}$-coordinates, $\mathcal{A}$-coordinates and cluster varieties.** We now describe the relation between $\mathcal{X}$-coordinates and $\mathcal{A}$-coordinates, and derive explicit formulas for the flip.

Recall that in Section 2.7 we expressed the cross-ratio of four framed Lagrangians in terms of the symplectic Λ-lengths (Lemma 2.20). Namely, let

Let $(L_i, \mathbf{v_i}) \in \mathrm{Lag}^{fr}(2n, \mathbb{R})$, with $i \in \{1, 2, 3, 4\}$, be four pairwise transverse framed Lagrangians. Then

$$[L_1, L_2, L_3, L_4]_{\mathbf{v_1}} = \Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21},$$

where $[L_1, L_2, L_3, L_4]_{\mathbf{v_1}}$ denotes the cross-ratio expressed in the basis $\mathbf{v_1}$.

Definition 8.11. We denote

$$CR_{1234} = -\Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21}$$

and call this expression the *cross ratio* in the basis $\mathbf{v_1}$.

For $\mathcal{A}$-coordinates the change of coordinates under a flip is given by the Exchange relation, see Proposition 2.17,

$$\Lambda_{24} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.$$

A nice formula for the flip in $\mathcal{X}$-coordinates can then be given by making a local change of coordinates.

Proposition 8.12. *Consider eight framed Lagrangians $(L_i, \mathbf{v_i})$, with $i \in \{1, \dots, 8\}$, as in Figure 8.1. To each inner edge of the 8-gon, we associate a cross ratio. We have the following formulas for the flip along the edge L_1L_2:*

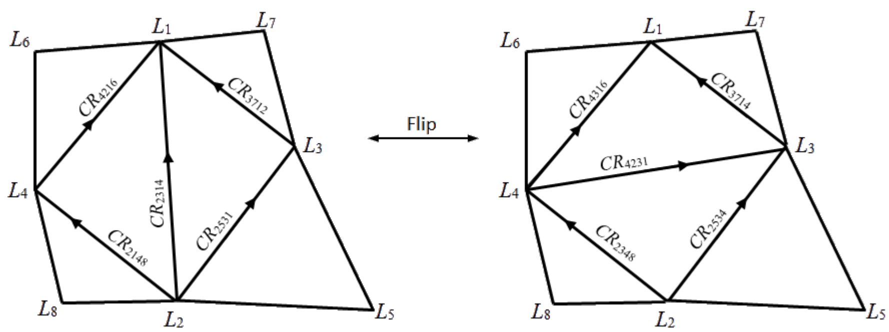

FIGURE 8.1.

$$\begin{aligned}
CR_{4231} &= \Lambda_{24}^{-1} \cdot CR_{2314}^{-T} \cdot \Lambda_{24} \\
CR_{2534} &= (\mathrm{Id} + CR_{2314})CR_{2531} \\
CR_{4316} &= CR_{4216}(\mathrm{Id} + CR_{4231}^{-1}) \\
CR_{2348} &= CR_{2148}(\mathrm{Id} + CR_{2314}^{-1})^{-1} \\
CR_{3714} &= (\mathrm{Id} + CR_{3142})^{-1}CR_{3712}
\end{aligned}$$

Note that in the case $n = 1$, these are precisely the formulas for the flip, see for example [10, Formula (1.30)], here we have a non-commutative generalization of them.

Proof.

$$CR_{2314} = -\Lambda_{42}^{-1}\Lambda_{41}\Lambda_{31}^{-1}\Lambda_{32}$$

If we do a flip, then we get

$$CR_{4231} = -\Lambda_{14}^{-1}\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{24} = [\Lambda_{ji} = -\Lambda_{ij}^{-T}] = \Lambda_{24}^{-1} \cdot CR_{2314}^{-T} \cdot \Lambda_{24}$$

For another cross ratio we have:

$$CR_{2531} = -\Lambda_{12}^{-1}\Lambda_{13}\Lambda_{53}^{-1}\Lambda_{52}$$

$$CR_{2534} = -\Lambda_{42}^{-1}\Lambda_{43}\Lambda_{53}^{-1}\Lambda_{52} = -\Lambda_{42}^{-1}(\Lambda_{41}\Lambda_{21}^{-1}\Lambda_{23} + \Lambda_{42}\Lambda_{12}^{-1}\Lambda_{13})\Lambda_{53}^{-1}\Lambda_{52} =$$
$$= -(\Lambda_{42}^{-1}\Lambda_{41} \cdot \Lambda_{21}^{-1}\Lambda_{23} \cdot \Lambda_{53}^{-1}\Lambda_{52} + (\Lambda_{42}^{-1}\Lambda_{42})\Lambda_{12}^{-1}\Lambda_{13}\Lambda_{53}^{-1}\Lambda_{52}) =$$
$$= CR_{2531} - \Lambda_{42}^{-1}\Lambda_{41}(\Lambda_{31}^{-1}\Lambda_{32}\Lambda_{32}^{-1}\Lambda_{31})\Lambda_{21}^{-1}\Lambda_{23}(\Lambda_{13}^{-1}\Lambda_{12}\Lambda_{12}^{-1}\Lambda_{13})\Lambda_{53}^{-1}\Lambda_{52} =$$
$$= CR_{2531} - (\Lambda_{42}^{-1}\Lambda_{41}\Lambda_{31}^{-1}\Lambda_{32})[\Lambda_{32}^{-1}\Lambda_{31}\Lambda_{21}^{-1}\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12}](\Lambda_{12}^{-1}\Lambda_{13}\Lambda_{53}^{-1}\Lambda_{52}) =$$
$$= CR_{2531} + CR_{2314}CR_{2531}$$

The proof for other cross ratios is similar. □

Appendix A. Spectral theorem with signature

A.1. **About correspondence between pairs of bilinear forms and symmetric linear maps.** Let V be n-dimensional vector space over some field K, b_1, b_2 be symmetric non-degenerate bilinear forms. We consider these forms as linear isomorphisms $b_i\colon V \to V^*$. Then we can consider the linear isomorphism $f\colon V \to V$ such that $f := b_1^{-1} \circ b_2$.

Lemma A.1. *The map f is symmetric with respect to the form b_1 and for all $x, y \in V$*

$$b_2(x, y) = b_1(x, fy)$$

Proof. We denote by capital letters matrices of corresponding tensors with respect to some fixed basis in V.

$$b_1(x, fy) = X^T B_1(FY) = X^T B_1(B_1^{-1}B_2Y) = X^T B_2 Y = b_2(x, y)$$
$$b_1(fx, y) = (FX)^T B_1 Y = (B_1^{-1}B_2X)^T B_1 Y = X^T B_2 Y =$$
$$= b_2(x, y) = b_1(x, fy)$$

$B_i^T = B_i$, $i = 1, 2$ because the forms are symmetric. □

A.1.1. *Definite pairs.*

Definition A.2. We say that the pair of forms (b_1, b_2) is *definite* if they are simultaneously diagonalizable i.e. there exist a basis of V in which both forms are diagonal.

Lemma A.3. *Pair (b_1, b_2) is definite if and only if f is diagonalizable. In this case basis can be chosen so that in this basis all three tensors b_1, b_2, f have diagonal matrices.*

Proof. ($\Rightarrow$) If (b_1, b_2) is definite then $f = b_1^{-1} \circ b_2$ is diagonal because b_1, b_2 are diagonal in chosen basis of V and corresponding dual basis of V^*.

($\Leftarrow$) Let $\mathbf{e} = (e_1, \ldots, e_n)$ some basis of V such that $f(e_i) = a_i e_i$.

$$a_j b_1(e_i, e_j) = b_1(e_i, fe_j) = b_1(fe_i, e_j) = a_i b_1(e_i, e_j)$$

If $a_i \neq a_j$ then $b(e_i, e_j) = 0$. If $a_i = a_j = a \in K$ then we consider the maximal subspace V_a such that $fw = aw$ for all $w \in V_a$. $V_a = Span(e_i \mid fe_i = ae_i)$, $I := \{i \mid fe_i = ae_i\}$. We can diagonalize b_1 on W. Without loss of generality we can assume that $(e_i \mid i \in I)$ is exactly this basis of W. We do this for all $a \in Spec(f)$.

So we get a basis $(e_i \mid i \in \{1,\dots,n\})$ such that $b_1(e_i,e_j)=0$ for $i\neq j$ and $fe_i = a_i e_i$. Therefore,

$$b_2(e_i,e_j) = b_1(fe_i,e_j) = a_i b_1(e_i,e_j) = 0$$

for $i \neq j$ □

Corollary A.4. *If $K=\mathbb{R}$ then we can choose a basis $\mathbf{e}$ so that $[b_1]_{\mathbf{e}} = I_{pq}$ where (p,q) is signature of the form b_1.*

A.1.2. *Definite pairs over $\mathbb{R}$.*

Corollary A.5. *If $K=\mathbb{R}$ and the pair (b_1,b_2) is definite and $[b_1]_e = I_{pq}$ in some basis $\mathbf{e}$ then there exists a (p,q)-orthogonal change-of-basis matrix $T\in O(p,q)$ which puts f to diagonal form, i.e. $[f]_{\mathbf{e}'} = T^{-1}FT$ is diagonal in the basis $\mathbf{e}' = \mathbf{e}T$ and $[b_1]_{\mathbf{e}'} = T^T[b_1]_{\mathbf{e}}T = [b_1]_{\mathbf{e}} = I_{pq}$.*

So we have showed that a definite pair of *real* bilinear forms is determined up to change of basis by a signature (p,q) of the first form and a diagonal matrix F, i.e. there exists a basis $\mathbf{e}$ so that

$$[b_1]_{\mathbf{e}} = I_{pq},\ [b_2]_{\mathbf{e}} = I_{pq}F$$

If such a basis is found, then we will say that the pair of forms is in standard form. It is easy to see that this basis is uniquely defined up to transformation $T\in \mathrm{O}(p,q)\cap \mathrm{O}(I_{pq}F)$.

Since F is diagonal and non-degenerate, we can consider the coordinate-wise square root

$$\Phi := \sqrt{|F|^{-1}}.$$

So we get that this change-of-basis matrix puts $I_{pq}F$ to $I_{p'q'}$: $I_{p'q'} = \Phi^T F\Phi$

A.1.3. *Non-definite pairs over $\mathbb{R}$ with real eigenvalues. Non-definite pairs over algebraic closed fields. Jordan blocks.* Now we want to find some analog of standard form for non-definite pairs.

Lemma A.6 (Jordan block over $\mathbb{R}$)**.** *Let $[f]_{\mathbf{e}} = J$ be a Jordan block with eigenvalue λ in some basis $\mathbf{e}$ of V. Then there exists another basis $\mathbf{e}'$ of V such that $[f]_{\mathbf{e}'} = J$ and either $[b_1]_{\mathbf{e}'} = C_n$ or $[b_1]_{\mathbf{e}'} = -C_n$ where*

$$C_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0\end{pmatrix}$$

Proof (Only idea). First, it is easy to see that $J^TB_1 = B_1J$ imply that all elements of $B_1 = (b_{ij})$ over the counterdiagonal are zero and, moreover, the matrix B_1 has a form

$$B_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & a_1\\ 0 & 0 & \dots & a_1 & a_2\\ & & \dots & & \\ 0 & a_1 & \dots & a_{n-2} & a_{n-1}\\ a_1 & a_2 & \dots & a_{n-1} & a_n\end{pmatrix}$$

Then we can rescale the basis such that $a_1 = \operatorname{sgn}(a_1)$. Then we can successively correct the basis using the following change-of-basis matrices

$$\begin{pmatrix} 1 & b & 0 & \dots & 0 & 0 \\ 0 & 1 & b & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 1 & b \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & b & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & b \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \dots,$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & b & 0 \\ 0 & 1 & 0 & \dots & 0 & b \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & b \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

for appropriate b to "kill" a_i for $i > 1$ and so get a form for B_1 as in lemma. □

Corollary A.7 (Jordan block over algebraic closed field)**.** *Over algebraic closed fields the basis can be chosen (by rescaling by i) so that*

$$C_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Lemma A.8 (Over $\mathbb{R}$ or algebraic closed field)**.** *The basis which was found in the previous lemma (in this lemma denoted by e) is unique up to multiplication of all vectors with* ± 1*.*

Proof. Let $\mathbf{u} = (u_i)$ be another basis with necessary property.

Step 1. By induction we will show that

$$u_k = \sum_{i=1}^{k} c_{k-i+1} e_i$$

1. $f(u_1) = \lambda u_1$, u_1 is an eigenvector of f. But all eigenvectors of f are ce_1, $c \in \mathbb{R}$. Therefore, $u_1 = c_1 e_1$ for some $c_1 \neq 0$.

2. We assume that $u_s = \sum_{i=1}^{s} c_{s-i+1} e_i$ for all $s < k$. $f(u_k) = au_k + u_{k-1}$, therefore $g(u_k) = f(u_k) - au_k \in \mathbb{R}u_{k-1} \leq \langle e_1, \dots, e_{k-1} \rangle$. If we assume

$$u_k = \sum_{i=1}^{n} c_{ki} e_i,$$

then

$$g(u_k) = \sum_{i=2}^{n} c_{ki} e_{i-1} \in \langle e_1, \dots, e_{k-1} \rangle .$$

Therefore $c_{ki} = 0$ for all $i > k$. Moreover

$$g(u_k) = u_{k-1} = \sum_{j=1}^{k-1} c_{k-1-j+1} e_j = [above] = \sum_{i=2}^{k} c_{ki} e_{i-1}$$

Therefore, $c_{ki} = c_{k-i+1}$, and so we have

$$u_k = \sum_{i=1}^{k} c_{k-i+1} e_i$$

Step 2. Now we show that $c_1 = \pm 1$ and $c_i = 0$ for $i > 1$. To do that we use the form b_1. By assumption

$$b_1(u_i, u_j) = b_1(e_i, e_j) = \delta_{i+j,n+1}$$

$$b_1(u_k, u_l) = \sum_{i=1}^{k} \sum_{j=1}^{l} c_{k-i+1} c_{l-j+1} b(e_i, e_j) = \sum_{i=1}^{k} c_{k-i+1} c_{l-n-1+i+1}$$

We assume here $c_i = 0$ for $i \leq 0$. If we take $l = n$, then we get

$$b_1(u_k, u_n) = \sum_{i=1}^{k} c_{k-i+1} c_i$$

For $k = 1$:

$$1 = b_1(u_1, u_n) = c_1 c_1$$

Therefore, $c_1 = \pm 1$. Further, we take $k = 2$,

$$0 = b_1(u_2, u_n) = c_2 c_1 + c_1 c_2$$

Therefore, $c_2 = 0$. And so on by induction, we assume $c_i = 0$ for all $1 < i < k$ for some k, then

$$0 = b_1(u_k, u_n) = c_k c_1 + c_{k-1} c_2 + \cdots + c_1 c_k$$

Therefore $c_k = 0$ for all $k \neq 1$. □

Corollary A.9.

$$\mathrm{O}(C_n) \cap \mathrm{O}(C_n J_n) = \{\pm \mathrm{Id}_n\}$$

Let b_1, b_2 be two bilinear forms in some vector space V of dimension n over $\mathbb{R}$ such that $[b_1]_{\mathbf{e}} = wC = wC_n$, $[b_2]_{\mathbf{e}} = CJ$ in some basis $\mathbf{e}$, where J is a Jordan block with eigenvalue l, $w = \pm 1$.

We want to find another basis $\mathbf{v}$ of V such that

$$[b_1^*]_{\mathbf{v}^*} = \mathrm{sgn}(lw)[b_2]_{\mathbf{e}}$$

$$[b_2^*]_{\mathbf{v}^*} = \mathrm{sgn}(lw)[b_1]_{\mathbf{e}}$$

If we express these conditions in matrix form then we need some matrix Φ such that (note that $C = C^{-1}$)

$$\mathrm{sgn}(lw)\Phi C \Phi^T = CJ$$

$$\mathrm{sgn}(lw)\Phi (CJ)^{-1} \Phi^T = C$$

Lemma A.10. $\Phi = \pm\Phi^T$

Proof. We assume $lw > 0$. The case $lw < 0$ is similar.

$$\Phi C \Phi^T = CJ$$
$$\Phi(CJ)^{-1}\Phi^T = C$$

are equivalent to

$$\Phi C \Phi^T = CJ$$
$$\Phi^T C \Phi = CJ$$

Therefore,

$$\Phi\Phi^{-T} C \Phi^{-1}\Phi^T = C$$
$$\Phi\Phi^{-T}(CJ)\Phi^{-1}\Phi^T = CJ$$

So $\Phi^{-1}\Phi^T \in O(C) \cap O(CJ) = \{\pm \mathrm{Id}\}$ (A.9) and we have $\Phi = \pm\Phi^T$. $\square$

Lemma A.11. *If there exists $\Phi \in \mathrm{Sym}(n, \mathbb{K})$ such that*

$$\mathrm{sgn}(lw)\Phi C \Phi = CJ$$

then this Φ is unique up to sign.

Proof. We assume $lw > 0$. The case $lw < 0$ is similar. Assume, there are two $\Phi, \Psi \in \mathrm{Sym}(n, \mathbb{K})$ such that

$$\Phi C \Phi = \Psi C \Psi = CJ$$

Then we have

$$\Psi\Phi^{-1} C \Phi^{-1}\Psi = C$$
$$\Psi\Phi^{-1}(CJ)\Phi^{-1}\Psi = CJ$$

So $\Phi^{-1}\Psi \in O(C) \cap O(CJ) = \{\pm \mathrm{Id}\}$ and we have $\Phi = \pm\Psi$. $\square$

Lemma A.12. *There exists $\Phi \in \mathrm{Sym}(n, \mathbb{K})$ such that*

$$\mathrm{sgn}(lw)\Phi C \Phi = CJ$$

$$\Phi = \pm \begin{pmatrix} 0 & 0 & \dots & 0 & \sqrt{|l|} \\ 0 & 0 & \dots & \sqrt{|l|} & x_1 \\ 0 & 0 & \dots & x_1 & x_2 \\ & & \dots & & \\ 0 & \sqrt{|l|} & \dots & x_{n-2} & x_{n-1} \\ \sqrt{|l|} & x_1 & \dots & x_{n-1} & x_n \end{pmatrix}$$

where x_i are some rational functions in $\sqrt{|l|}$.

Proof. Put this matrix in the equation $\Phi C \Phi = CJ$ and calculate successively all coefficients. $\square$

Remark A.13. Φ is defined up to sign. To make the choice of Φ unique, we take plus sign in case $l > 0$ or $w > 0$. Otherwise, we take minus sign. At this point, it does not really matter how we choose the sign. It will be important later when we will consider degenerate representations.

A.2. **About classification of symmetric maps.**

A.2.1. *Over algebraic closed fields.* In this section we want to show that over algebraic closed field K for every symmetric (with respect to some non-degenerate form b) linear map f there is an orthogonal basis $\mathbf{e}$ such that

$$[f]_{\mathbf{e}} = \begin{pmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_k \end{pmatrix},$$

where J_k is a $n_k \times n_k$ Jordan block corresponding to the eigenvalue λ_k and

$$[b]_{\mathbf{e}} = \begin{pmatrix} I_1^* & 0 & \dots & 0 & 0 \\ 0 & I_2^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_k^* \end{pmatrix},$$

where

$$I_s^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_s \times n_s}.$$

By the theorem of Jordan we already know that there exists a basis $\mathbf{e}$ such that $[f]_{\mathbf{e}}$ has a necessary form. We want to show that we can correct this basis to another basis $\mathbf{e}'$ such that $[b]_{\mathbf{e}'}$ is of the form as above and $[f]_{\mathbf{e}} = [f]_{\mathbf{e}'}$.

Lemma A.14. *Blocks with different eigenvalues are orthogonal.*

Proof. The proof is identically to the proof of lemma A.3 for eigenvectors. We have just to do the same by induction for all pairs of generalized eigenvectors of corresponding blocks. □

As we also have seen, if we restrict the form b to each Jordan block, then, if this form is not degenerate, then the basis of this block can be chosen so that this restriction of b has a form:

$$I^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Such blocks we will call “non-degenerate”.

Therefore, we have to prove two things:

1. If the restriction of b to some block is degenerate, then there exists another block with the same eigenvalue. Using this block we will correct the “degenerate” block to “non-degenerate” block.

2. We can orthogonalize non-degenerate blocks with the same eigenvalue.

Lemma A.15. *Let J_s be a block, $m := n_s = \dim(J_s)$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of $\mathbf{e}$ for this block, $V = \mathrm{Span}(\mathbf{v})$.*

Let J_p be a block with the same eigenvalue λ, $l := n_p = \dim(J_p)$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$.

If $m > l$, then

$$B_{\mathbf{v},\mathbf{w}} := (b(v_i, w_j)) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & c_1 & c_2 \\ & & \dots & & \\ c_1 & c_2 & \dots & c_{n-1} & c_n \end{pmatrix}.$$

If $m < l$, then

$$B_{\mathbf{v},\mathbf{w}} := (b(v_i, w_j)) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & 0 & 0 & \dots & c_1 & c_2 \\ & & \dots & & & \dots & & \\ 0 & 0 & \dots & c_1 & c_2 & \dots & c_{m-1} & c_m \end{pmatrix}.$$

Proof. We proof the first case. The second is analogous. We use

$$b(f(v_i), w_j) = b(v_i, f(w_j)) = \lambda b(v_i, w_j) + b(v_i, w_{j-1})$$
$$b(f(v_i), w_j) = \lambda b(v_i, w_j) + b(v_{i-1}, w_j)$$

We get $b(v_i, w_{j-1}) = b(v_{i-1}, w_j)$ for all $i = 1, ..m$, $j = 1, \dots, l$ and $b(v_j, w_1) = 0$ for all $j = 1, \dots, m$. So we get $(b(v_i, w_j))$ as above inductively. $\square$

Lemma A.16. *Let J_s be a block, $m := n_s$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of $\mathbf{e}$ for this block, $V = \mathrm{Span}(\mathbf{v})$. Let J_p be a block with the same eigenvalue λ, $l := n_p$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$.*

Then $\mathbf{u} = \mathbf{v} + \mathbf{w}T$ is a basis of Jordan block with the same eigenvalue λ if and only if T has the following form: for $m \geq l$

$$T = \begin{pmatrix} c_1 & c_2 & \dots & c_{m-1} & c_m \\ 0 & c_1 & \dots & c_{m-2} & c_{m-1} \\ & & \dots & & \\ 0 & 0 & \dots & c_1 & c_2 \\ 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

for $m \leq l$

$$T = \begin{pmatrix} 0 & \dots & 0 & c_1 & c_2 & \dots & c_{l-1} & c_l \\ 0 & \dots & 0 & 0 & c_1 & \dots & c_{l-2} & c_{l-1} \\ & & & & & \dots & & \\ 0 & \dots & 0 & 0 & 0 & \dots & c_1 & c_2 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & c_1 \end{pmatrix}$$

Matrices of this form we will call diagonal upper triangular.

Proof. For every basis $\mathbf{u} = (u_1, \dots, u_s)$ we denote by $\partial\mathbf{u} := (0, u_1, \dots, u_{s-1})$. Then for each basis of Jordan block we have $f(\mathbf{u}) = \lambda\mathbf{u} + \partial\mathbf{u}$. The map ∂ in basis $\mathbf{u}$ is given by matrix

$$P := [\partial]_{\mathbf{u}} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Now we want $f(\mathbf{u}) = \lambda\mathbf{u} + \partial\mathbf{u}$ for $\mathbf{u} = \mathbf{v} + \mathbf{w}T$. That means

$$f(\mathbf{v}+\mathbf{w}T) = f(\mathbf{v})+f(\mathbf{w})T = \lambda\mathbf{v}+\partial\mathbf{v}+(\lambda\mathbf{w}+\partial\mathbf{w})T = \lambda(\mathbf{v}+\mathbf{w}T)+\partial\mathbf{v}+(\partial\mathbf{w})T =$$

$$= \lambda\mathbf{u} + \partial\mathbf{u} + (\partial\mathbf{w})T - \partial(\mathbf{w}T).$$

That means, $\mathbf{u}$ is Jordan basis if and only if $PT = TP$.

If $T = (t_{ij})$ then $PT = (t_{i-1,j})$, $TP = (t_{i,j+1})$ (to make this notation completely correct, we assume here $t_{ij} = 0$ for $i > l$ or $j > m$ or for $i, j < 1$). That means, $t_{i-1,j} = t_{i,j+1}$ and $t_{1j} = 0$ for $j > 1$, $t_{im} = 0$ for $i < m$. Therefore, T has a necessary form. □

Lemma A.17. *Let J_s be a block, $m := n_s$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of $\mathbf{e}$ for this block, $V = \mathrm{Span}(\mathbf{v})$. Let b_V be degenerate. Then there exists another block J_p with the same eigenvalue λ, $l := n_p$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$ and $b_{V\oplus W}$ is not degenerate.*

Moreover, there exists another basis $\mathbf{u} = (u_1, \dots, u_m)$ such that $U = \mathrm{Span}(\mathbf{u})$ is invariant by f, $U \oplus W = V \oplus W$, $[f|_U]_{\mathbf{u}} = J_s$ and b_U is not degenerate.

Proof. Because of

$$b(f(v_i), v_j) = b(v_i, f(v_j)) = \lambda b(v_i, v_j) + b(v_i, v_{j-1})$$

$$b(f(v_i), v_j) = \lambda b(v_i, v_j) + b(v_{i-1}, v_j)$$

we get $b(v_i, v_{j-1}) = b(v_{i-1}, v_j)$ for all $i, j = 1, \dots, m$ and $b(v_1, v_j) = 0$ for all $j = 1, \dots, m-1$. Therefore,

$$B_{\mathbf{v}} := [b|_V]\mathbf{v} = \begin{pmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_1 & a_2 \\ & & \dots & & \\ 0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}$$

This matrix is degenerate, that means that $a_1 = 0$ and v_1 is orthogonal to the whole block. v_1 is also orthogonal to all blocks with eigenvalues different form λ. But the form b is not degenerate. Therefore, there exists another block J_p with the eigenvalue λ and basis $\mathbf{w} = (w_1, \dots, w_l)$ and $b(v_1, w_l) \neq 0$ because by the lemma A.15 $b(v_1, w_r) \neq 0$ for all $r < l$. We can assume that $m \geq l$, otherwise it is easy to see by considering of determinant of b that the form is degenerate on the whole space.

Let $\mathbf{u} = \mathbf{v} + \mathbf{w}T$ for some diagonal upper triangular T. Then $b|_U$ is non-degenerate if and only if $b(u_1, u_m) \neq 0$. $b(u_1, u_m) = b(v_1, v_m) + 2b(v_1, (wT)_m) + b((\mathbf{w}T)_1, (\mathbf{w}T)_m) = 0 + 2(B_{\mathbf{v},\mathbf{w}}T)_{1m} + (T^T B_{\mathbf{w}} T)_{1m} \neq 0$

If $m > l$ or $m = l$ and $B_{\mathbf{w}}$ degenerate, then $T^T B_{\mathbf{w}} T$ is also degenerate and $(T^T B_{\mathbf{w}} T)_{1m} = 0$. In this case we can $T = (0|\,\mathrm{Id})$. Then $(B_{\mathbf{v},\mathbf{w}}T)_{1m} = b(v_1, w_l) \neq 0$.

Otherwise, we can take $T = c\,\mathrm{Id}$ then

$$2(B_{\mathbf{v},\mathbf{w}}T)_{1m} + (T^T B_{\mathbf{w}} T)_{1m} = cb(v_1, w_m) + c^2 b(w_1, w_m) \neq 0$$

for $c \neq -\frac{b(v_1,w_m)}{b(w_1,w_m)}$.

□

Using last lemma we can always assume that the basis is chosen so that all Jordan blocks are non degenerate with respect to b. Now we want to correct this basis so that different blocks are orthogonal.

Lemma A.18. *Let J_s is a non degenerate with respect to b Jordan block, $m := n_s$, λ is its eigenvalue. $\mathbf{v} = (v_1, \dots, v_m)$ is corresponding subbasis of $\mathbf{e}$ for this block, $V = \mathrm{Span}(\mathbf{v})$.*

Let J_p be another block with the same eigenvalue λ, $l := n_p$, $\mathbf{w} = (w_1, \dots, w_l)$ corresponding subbasis of $\mathbf{e}$, $W = \mathrm{Span}(\mathbf{w})$. We assume $m \geq l$.

Then there exists a diagonal upper triangular matrix T such that $\mathbf{u} = \mathbf{w} + \mathbf{v}T$ is a basis of Jordan block which is orthogonal to J_s and $V \oplus W = U \oplus W$

Proof. That $\mathbf{u} = \mathbf{w} + \mathbf{v}T$ is a basis of Jordan block, we already know by the lemma A.16. We want orthogonality. That means

$$0 = b(\mathbf{v}, \mathbf{w} + \mathbf{v}T) = b(\mathbf{v}, \mathbf{w}) + b(\mathbf{v}, \mathbf{v}T) = B_{\mathbf{v},\mathbf{w}} + B_{\mathbf{v}}T$$

Because $B_{\mathbf{v}}$ is not degenerate, we have

$$T = B_{\mathbf{v},\mathbf{w}} B_{\mathbf{v}}^{-1}$$

This is a product of two diagonal upper triangular matrices, which is diagonal upper triangular. The new block is not degenerate because, otherwise, the form would be degenerate on $V \oplus U$, but this is not the case. □

Corollary A.19. *If we have many blocks with the same eigenvalue, then we do the process as in previous lemma successively as in Gram–Schmidt orthogonalization.*

A.2.2. *Case $K = \mathbb{R}$ with real eigenvalues.* Because $\mathbb{R}$ is not algebraic closed, we have to take care by the process which we did in the case K algebraic closed.

First, we have to assume that all eigenvalues of f are real, otherwise the theorem of Jordan does not guarantee us that the Jordan basis exists.

Theorem A.20. *For every symmetric with respect to some non-degenerate form b linear map f with real eigenvalues there is an orthogonal basis $\mathbf{e}$ such* *that*

$$[f]_{\mathbf{e}} = \begin{pmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_k \end{pmatrix}$$

where J_k is a $n_k \times n_k$ Jordan block corresponding to the eigenvalue λ_k and

$$[b]_{\mathbf{e}} = \begin{pmatrix} \sigma_1 I_1^* & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 I_2^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & \sigma_k I_k^* \end{pmatrix}$$

where $\sigma_i = \pm 1$ and

$$I_s^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \dots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_s \times n_s}$$

Moreover,

$$\operatorname{sgn}(I_s^*) = \begin{cases} 0 & , \textit{for } n_s \textit{ even} \\ 1 & , \textit{for } n_s \textit{ odd} \end{cases}$$

and, therefore,

$$\operatorname{sgn}(b) = \sum_{i=1}^{k} \sigma_i \operatorname{sgn}(I_i^*) = \sum_{\{i \mid n_i \textit{ is odd}\}} \sigma_i$$

Proof. In this case, we only have to prove that in each Jordan block the basis can be chosen so that the restriction of b on this block is represented by a matrix $\pm I^*$.

To do this, first, we consider a complexification of f and find a complex basis $(v_i)_{i=1}^n$ for a fixed chosen Jordan block $n \times n$ as in the previous section. That means

$$f(v_i) = \lambda v_i + v_{i+1}$$
$$b(v_i, v_j) = \delta_{i+j,n+1}.$$

If we conjugate these equalities, we get (since $\lambda \in \mathbb{R}$):

$$f(\bar{v}_i) = \lambda \bar{v}_i + \bar{v}_{i+1}$$
$$b(\bar{v}_i, \bar{v}_j) = \delta_{i+j,n+1}.$$

Case 1. (v_i) and $(\bar{v}_i)$ define bases of different complex Jordan blocks. Therefore v_i and $\bar{v}_i$ are not collinear and there exist unique collections of non-zero vectors (u_i), (w_i) such that

$$v_i = \frac{u_i + i w_i}{\sqrt{2}}.$$

Therefore, (u_i) and (w_i) define reals bases of two different Jordan blocks. We can correct these bases so that they are orthogonal and the restriction of b on corresponding subspaces is represented by a matrix $\pm I^*$ [see lemma A.6].

Case 2. (v_i) and $(\bar{v}_i)$ define bases of the same complex Jordan block. Because of uniqueness of basis $\bar{v}_i = \pm v_i$.

Case 2.1. $v_i = \bar{v}_i$. That means, $\mathbf{v} = (v_i)$ is a real basis of the chosen Jordan block with $[b|_{\mathrm{Span}_{\mathbb{R}}(\mathbf{v})}]_{\mathbf{v}} = I^*$.

Case 2.2. $v_i = -\bar{v}_i = iw_i$. That means, $\mathbf{w} = (w_i)$ is a real basis basis of the chosen Jordan block with $[b|_{\mathrm{Span}_{\mathbb{R}}(\mathbf{w})}]_{\mathbf{w}} = -I^*$. □

A.2.3. *Case $K = \mathbb{R}$ with complex eigenvalues. Generalized Jordan blocks.*

Remark A.21. For some technical reasons, we need some linear order on $\mathbb{C}$. It does not really matter which one, but to make some constructions unique we have to fix one. We will use the following order: we say $z > z'$ if $\mathrm{Re}(z) > \mathrm{Re}(z')$ or $\mathrm{Re}(z) = \mathrm{Re}(z')$ and $\mathrm{Im}(z) > \mathrm{Im}(z')$.

If the linear map f have a complex not real eigenvalue $\lambda = a + ib$ then it has an eigenvalue $\bar{\lambda} = a - ib$ as well because the characteristic polynomial is real. We consider some Jordan block J with eigenvalue λ of the size $m \times m$. Then we have automatically a Jordan block for $\bar{\lambda}$. Moreover, these both blocks have the same size because if

$$f(v_j) = \lambda v_j + v_{j-1}$$

then

$$f(\bar{v}_j) = \bar{\lambda}\bar{v}_j + \bar{v}_{j-1}$$

where (v_j) is a basis of the block J. So $(\bar{v}_j)$ is a basis of another Jordan block with eigenvalue $\bar{\lambda}$ which we denote by $\bar{J}$. We denote

$$v_j = \frac{u_j + iw_j}{\sqrt{2}}$$

We can also assume $b(v_j, v_k) = b(\bar{v}_j, \bar{v}_k) = \delta_{j+k,m+1}$

We consider another basis for pair of blocks $(J, \bar{J})$:

$$u_j = \frac{v_j + \bar{v}_j}{\sqrt{2}}, \quad w_j = \frac{v_j - \bar{v}_j}{i\sqrt{2}}$$

It is easy to see that

$$f(u_j) = au_j - bw_j + u_{j-1}$$

$$f(w_j) = bu_j + aw_j + w_{j-1}$$

Because of the discussion above we can assume that all complex Jordan blocks are orthogonal to each other. Therefore,

$$b(u_j, u_k) = -b(w_j, w_k) = b(v_j, v_k) = \delta_{j+k,m+1}$$

So we get that in the real basis $(u_1, w_1, \ldots, u_m, w_m)$ the pair of blocks $(J, \bar{J})$ is represented by the following matrix

$$K = \begin{pmatrix} a & b & 1 & 0 & \ldots & 0 & 0 \\ -b & a & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & a & b & \ldots & 0 & 0 \\ 0 & 0 & -b & a & \ldots & 0 & 0 \\ & & & & \ldots & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & a & b \\ 0 & 0 & 0 & 0 & \ldots & -b & a \end{pmatrix}_{2m \times 2m}$$

which we will call generalized Jordan block. The restriction of b on $\mathrm{Span}(u_1, w_1, \ldots, u_m, w_m)$ have the form I^{2*}_{2m}, where

$$I^{2*}_{2m} = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 \\ & & & & \ldots & & \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & -1 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{2m \times 2m}$$

This matrix has signature

$$\mathrm{sgn}(I^{2*}_{2m}) = 0$$

Moreover, because all complex Jordan blocks are orthogonal, this generalized block is orthogonal to other blocks.

Corollary A.22. *If f consists only on one generalized Jordan block then the basis above is unique up to simultaneous multiplication of all basis vectors with -1. The proof is identical to the proof of A.8*

Lemma A.23. *There exists unique up to sign $\Phi \in \mathrm{Sym}(n, \mathbb{R})$ such that*

$$\Phi I^{2*} \Phi = I^{2*} K$$

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & c & d \\ 0 & 0 & 0 & 0 & \ldots & d & -c \\ 0 & 0 & 0 & 0 & \ldots & * & * \\ 0 & 0 & 0 & 0 & \ldots & * & * \\ & & & & \ldots & & \\ 0 & 0 & c & d & \ldots & * & * \\ 0 & 0 & d & -c & \ldots & * & * \\ c & d & * & * & \ldots & * & * \\ d & -c & * & * & \ldots & * & * \end{pmatrix}$$

where $(c + id)^2 = a + ib$ and $*$ are some rational functions in c, d.

Proof. Similar to A.12 □

Remark A.24. The pair (c, d) is defined up to sign. To make Φ unique we choose (c, d) so that $c + id$ is the biggest square root of $a + ib$.

A.3. **Standard form of a pair of bilinear forms.** So we can summarize that for each bilinear form b and each linear operator f which is symmetric with respect to b there exists a basis $\mathbf{e}$ such that

$$[b]_{\mathbf{e}} = \begin{pmatrix} \mathcal{I}_1^* & 0 & 0 \\ 0 & -\mathcal{I}_2^* & 0 \\ 0 & 0 & \mathcal{I}^{2*} \end{pmatrix}, \quad [f]_{\mathbf{e}} = \begin{pmatrix} \mathcal{J}_1 & 0 & 0 \\ 0 & \mathcal{J}_2 & 0 \\ 0 & 0 & \mathcal{K} \end{pmatrix}$$

where for $r = 1, 2$

$$\mathcal{I}_r^* = \begin{pmatrix} I_{1r}^* & 0 & \dots & 0 & 0 \\ 0 & I_{2r}^* & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_{k_r r}^* \end{pmatrix}, \quad \mathcal{J}_r = \begin{pmatrix} J_{1r} & 0 & \dots & 0 & 0 \\ 0 & J_{2r} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & J_{k_r r} \end{pmatrix}$$

$$\mathcal{I}^{2*} = \begin{pmatrix} I_1^{2*} & 0 & \dots & 0 & 0 \\ 0 & I_2^{2*} & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & I_s^{2*} \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} K_1 & 0 & \dots & 0 & 0 \\ 0 & K_2 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & K_s \end{pmatrix}$$

where $n_{ir} := \dim(I_{ir}^*) = \dim(J_{ir})$, $m_j := \dim(I_j^{2*}) = \dim(K_j)$.

Definition A.25 (Order on blocks)**.** For two (generalized) Jordan blocks J with eigenvalue l and J' with eigenvalue l' we will say that $J > J'$ if $\dim J > \dim J'$ or $\dim J = \dim J'$ and $l > l'$ (for generalized blocks we compare complex numbers using the order defined earlier).

Definition A.26 (Standard form of a pair of bilinear forms)**.** If the basis $\mathbf{e}$ is chosen as above and blocks in $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{K}$ are in order of decreasing then we will say that pair of forms $b_1 = b$ and $b_2 = b \circ f$ is in the *standard form.* We will use the following notation:

$$X(b_1, b_2) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{K})$$

$$X^0(b_1, b_2) = [f]_{\mathbf{e}}, \;\; X^1(b_1, b_2) = [b_1]_{\mathbf{e}}, \;\; X^2(b_1, b_2) = [b_2]_{\mathbf{e}}$$

Remark A.27. Because

$$\operatorname{sgn}(I^*) = \begin{cases} 0 & \text{, for } \dim I^* \text{ even} \\ 1 & \text{, for } \dim I^* \text{ odd} \end{cases}, \quad \operatorname{sgn}(I^{2*}) = 0$$

we get

$$\operatorname{sgn}(b) = \#\{i | \dim I_{i1}^* \text{ is odd}\} - \#\{i | \dim I_{i2}^* \text{ is odd}\}$$

Remark A.28. The standard form is unique. The basis, in which some pais of forms has standard form, is unique up to simultaneous sign change of basis vectors which correspond to the some (generalized) Jordan block if and only if all (generalized) blocks are distinct, i.e. if two blocks have the same eigenvalue then their sizes are different.

Remark A.29. $X(b_1, b_2)$ defines $X^0(b_1, b_2)$, $X^1(b_1, b_2)$, $X^2(b_1, b_2)$ uniquely and defines b_1, b_2 uniquely up to change of basis.

$$X^0(b_1, b_2) = \operatorname{diag}(X(b_1, b_2))$$

$$X^0(b_1, b_2) = (X^1(b_1, b_2))^{-1} X^2(b_1, b_2)$$

We define the signature

$$\operatorname{sgn}(X(b_1, b_2)) := \operatorname{sgn}(b_1)$$

A.4. **Back transformation.**

Definition A.30. We will say that matrix H is *consistent* to the pair of forms (b_1, b_2), if

$$H = \operatorname{diag}(H_1, H_2, H_3)$$

$$H_k = \operatorname{diag}(H_{1k}, \dots, H_{k_r k})$$

and $\dim H_{ij} = \dim J_{ij}$ for $j = 1, 2$, $\dim H_{i3} = \dim K_i$ for all possible i.

Definition A.31. Let $Y = \operatorname{diag}(Y_1, \dots, Y_s)$, $\sigma \in \operatorname{Sym}(\{1, \dots, s\})$. The matrix

$$T_\sigma = \begin{pmatrix} T_{11} & \dots & T_{1s} \\ & \dots & \\ T_{s1} & \dots & Tss \end{pmatrix}$$

is called *block permutation matrix* for Y if $T_{i,\sigma(i)} = \mathrm{Id}_{\dim Y_i}$ for all i and $T_{ij} = 0$ for all other (i, j).

Remark A.32. It is easy to see that

$$T_\sigma^T \operatorname{diag}(Z_1, \dots, Z_s) T = \operatorname{diag}(Z_{\sigma(1)}, \dots, Z_{\sigma(Z_s)})$$

for all $\operatorname{diag}(Z_1, \dots, Z_s)$ such that $\dim Z_i = \dim Y_i$ for all i.

Definition A.33 (Minimal ordering matrix)**.** Let

$$Y = \operatorname{diag}(Y_1, \dots, Y_s), \quad Z = \operatorname{diag}(Z_1, \dots, Z_s)$$

and $\dim Y_i = \dim Z_i$ for all $i \in \{1, \dots, s\}$. Moreover, assume

$$Y_i \in \{I_r^*, -I_r^*, I_r^{2*} \mid r \in \mathbb{N}\}$$

and Z_i is a Jordan block for all $i \in \{1, \dots, s\}$ such that $Y_i = \pm I_r^*$ and Z_i is a generalized Jordan block for all $i \in \{1, \dots, s\}$ such that $Y_i = I_r^{2*}$.

We will say that a block permutation matrix $T = T_\sigma$ for Y is *minimal ordering matrix* for (Y, Z) if

- $(Y', Y'Z') := (T^T Y T, T^T Y Z T)$ is the standard form for some pair of bilinear forms, where

$$Y' = \operatorname{diag}(Y_{\sigma(1)}, \dots, Y_{\sigma(s)})$$

$$Z' = \operatorname{diag}(Z_{\sigma(1)}, \dots, Z_{\sigma(s)})$$

- if $Y_i Z_i = Y_j Z_j$ for $i < j$ then $\sigma(i) < \sigma(j)$.

Remark A.34. For fixed pair (Y,Z) as above the minimal ordering matrix is unique because is well-defined by the corresponding permutation σ which is unique.

Proposition A.35. *There exist consistent to* (b_1,b_2) *matrix* $\Phi \in \mathrm{Sym}(n,\mathbb{R})$ *and the (unique) minimal ordering matrix* T *for* $(\Phi X^2(b_1,b_2)^{-1}\Phi, \Phi X^1(b_1,b_2)\Phi)$ *such that*

$$T^T \Phi X^1(b_1,b_2)\Phi T = X^2(b_2^*,b_1^*) =: \tilde{X}^2(b_1,b_2)$$

$$T^T \Phi X^2(b_1,b_2)^{-1}\Phi T = X^1(b_2^*,b_1^*) =: \tilde{X}^1(b_1,b_2).$$

We will call this transformation back transformation*.*

Proof. It follows from A.12 and A.23. We take $\Phi = \mathrm{diag}(\Phi_1,\ldots,\Phi_p)$ where Φ_i are from A.12 or A.23 for corresponding pair of blocks of $(X^1(b_1,b_2), X^2(b_1,b_2))$. After that we do a minimal ordering. □

Remark A.36. In the previous proposition, Φ is unique up to sign of each block. But as we already have seen, this sign can be chosen in a canonical way. So we can assume that Φ and T are well defined by (b_1,b_2).

Remark A.37. The direct calculation shows that the back transformation applied twice gives the identity map.

Corollary A.38. *The last proposition can be reformulated in the following way:*

Let (b_1,b_2) *is a pair of bilinear forms on a vector space* V *and in a basis* $\mathbf{e}$*:*

$$[b_1]_{\mathbf{e}} = X^1(b_1,b_2),\ [b_2]_{\mathbf{e}} = X^2(b_1,b_2)$$

We consider a pair of bilinear forms (b_2^*,b_1^*) *on the dual space* V^**. In the dual basis* f*:*

$$[b_1]_{\mathbf{f}} = X^1(b_1,b_2)^{-1},\ [b_2]_{\mathbf{f}} = X^2(b_1,b_2)^{-1}$$

The change of basis on V *given by a matrix* $\Phi^{-1}T^{-T}\colon \mathbf{e} \mapsto \mathbf{e}'$ *induce change of basis on* V^* *by a matrix* $\Phi T\colon \mathbf{f} \mapsto \mathbf{f}'$ *so that*

$$[b_1^*]_{\mathbf{f}'} = X^2(b_2^*,b_1^*),\ [b_2^*]_{\mathbf{f}'} = X^1(b_2^*,b_1^*)$$

This change-of-basis is determined by $X(b_1,b_2)$*. We denote this transformation on bases of* V *by* $\sigma_{X(b_1,b_2)}$*, the corresponding dual transformation of bases of* V^* *is denoted by* $\sigma^*_{X(b_1,b_2)}$*. This transformation will be used to define the basis associated to the opposite oriented edge.*

A.5. (p,q)**-shape transformation.** Let

$$\mathbf{n} := (n_1,\ldots,n_{k_1}),\ \ \mathbf{m} := (m_1,\ldots,m_{k_2}),\ \ \mathbf{r} := (r_1,\ldots,r_{k_3})$$

be three decreasing sequences of natural numbers.

$$I_{\mathbf{nmr}} := \mathrm{diag}(I^*_{n_1},\ldots,I^*_{n_{k_1}},-I^*_{m_1},\ldots,-I^*_{m_{k_2}},I^{2*}_{r_1},\ldots,I^{2*}_{r_{k_3}})$$

We consider this matrix as a matrix of some bilinear form. Let (p, q) be the signature of this form. We fix one matrix $P_{\mathbf{nmr}}$ such that

$$P_{\mathbf{nmr}}^T I_{pq} P_{\mathbf{nmr}} = I_{\mathbf{nmr}}$$

and the corresponding P-matrix for I_{pq} is Id.

Definition A.39. We denote by $\mathcal{P}_{pq}$ the set of all matrices $P_{\mathbf{nmr}}$ such that $I_{\mathbf{nmr}}$ has signature (p, q).

Definition A.40. Let (b_1, b_2) be a pair of bilinear forms and b_1 has signature (p, q). We denote by $P_{b_1 b_2}$ the corresponding $P_{\mathbf{nmr}}$ as above such that

$$X^1(b_1, b_2) = P_{b_1 b_2}^T I_{pq} P_{b_1 b_2}$$

Definition A.41. Let $X = X(b_1, b_2)$ for some pair of forms (b_1, b_2). Then b_1 has signature (p, q). We denote by P_X the corresponding $P_{b_1 b_2}$ as above such that

$$X^1 = X^1(b_1, b_2) = P_X^T I_{pq} P_X$$

Remark A.42. As we have seen before, $X(b_1, b_2)$ defines $X^0(b_1, b_2)$, $X^1(b_1, b_2)$, $X^2(b_1, b_2)$. So if we know $X(b_1, b_2)$, we do not need any information about (b_1, b_2). Therefore, sometimes we will write just X instead of $X(b_1, b_2)$ and also X^0, X^1, X^2 instead of $X^0(b_1, b_2)$, $X^1(b_1, b_2)$, $X^2(b_1, b_2)$ (and correspondent expressions with $\tilde{}$) if forms (b_1, b_2) are not important.

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